2017.12.17

Week #13 & #14

Frequency Response EE362

The Concept of Frequency Response

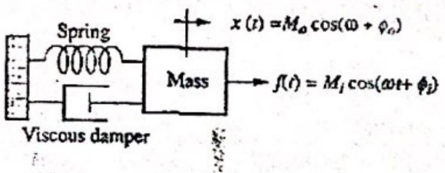
- In the steady state, sinusoidal inputs to a linear system generate sinusoidal responses of the same frequency. Even though these responses are of the same frequency as the input, they differ in **amplitude and phase angle** from the input.

- Sinusoids can be represented as complex numbers called **phasors**. The magnitude of the complex number is the amplitude of the sinusoid, and the angle of the complex number is the phase angle of the sinusoid.

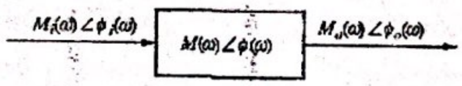
$$M_1 \cos(\omega t + \phi_1) \quad \text{represented as} \quad M_1\angle\phi_1$$

Consider the mechanical system of figure below. If the input force, f(t), is sinusoidal, the steady-state output response, x(t), of the system is also sinusoidal and at the same frequency as the input.

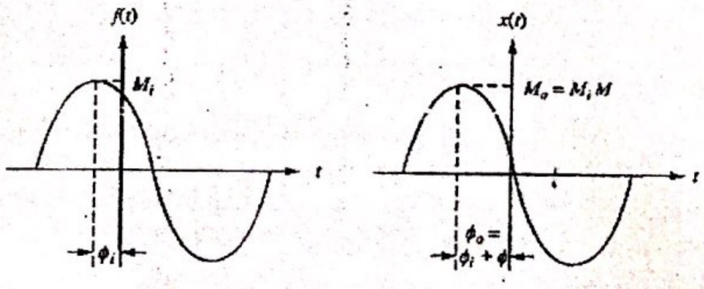
system


Spring
$x(t) = M_o \cos(\omega t + \phi_o)$
Mass
$f(t) = M_i \cos(\omega t + \phi_i)$
Viscous damper

transfer function;


$M_i(\omega)\angle\phi_i(\omega)$
$M(\omega)\angle\phi(\omega)$
$M_o(\omega)\angle\phi_o(\omega)$

input and output waveforms



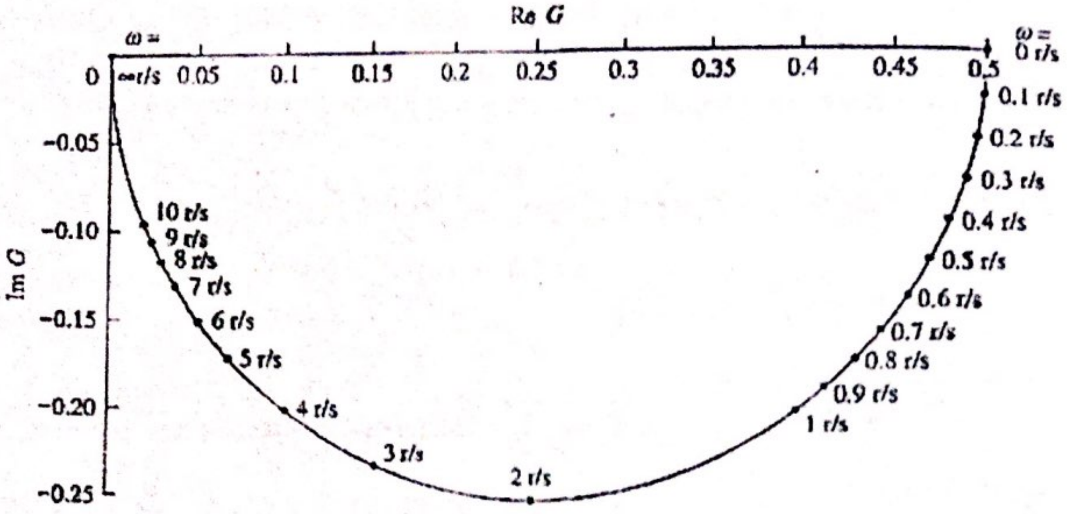
Input

Output

Thus, the steady-state output sinusoid is

52

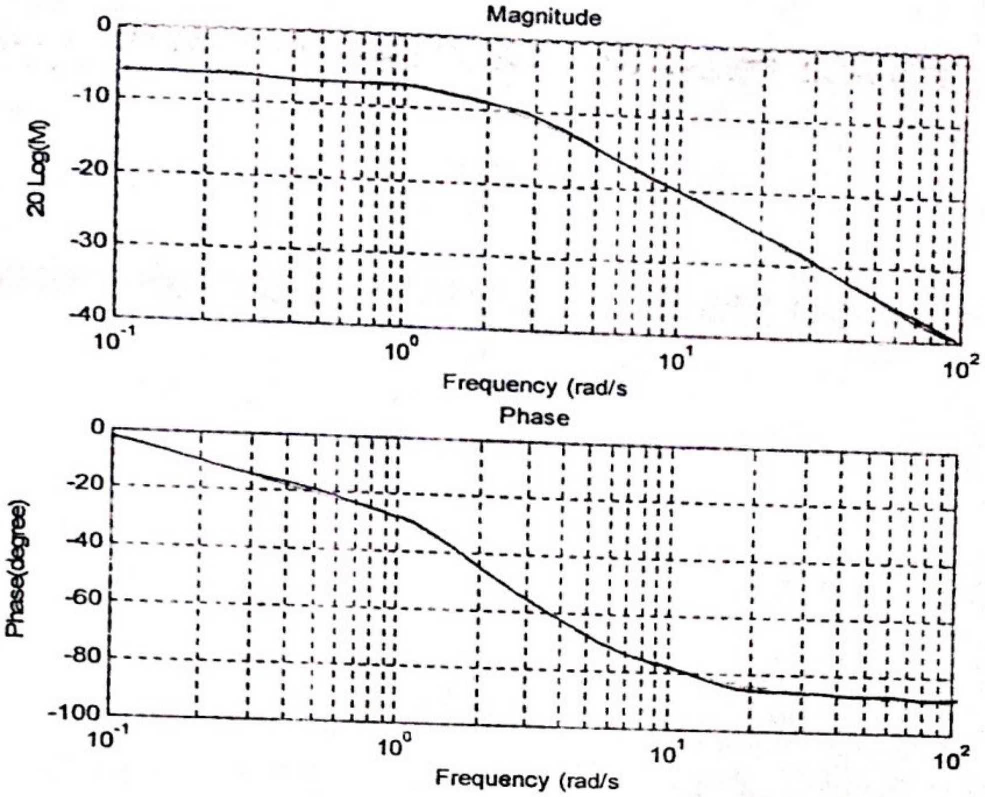
2- Polar plot

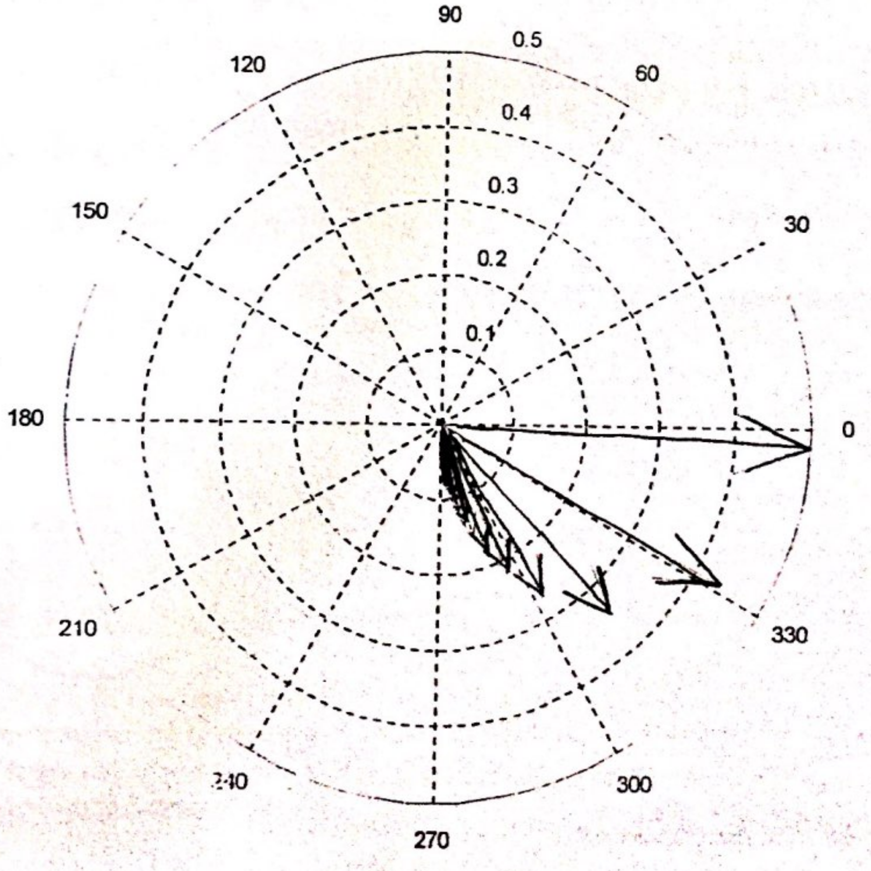


Note: r/s = rad/s

MATLAB code

```
syms s w
G= 1/(s+2);
pretty(G)

s= j*w;
G= subs(G,s);
w=[0.1:100];
G= subs(G,w);

M =abs(G)
Fi =rad2deg(angle(G))

M= 20*log10(M)
subplot(211),semilogx(w,M)
xlabel('Frequency (rad/s')
ylabel('20 Log(M)')
title('Magnitude')
grid
subplot(212),semilogx(w,Fi)
xlabel('Frequency (rad/s')
ylabel('Phase(degree)')
title('Phase')
grid
figure
compass(G)
```


Magnitude
20 Log(M)
0
-10
-20
-30
-40
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (rad/s
Phase
Phase(degree)
0
-20
-40
-60
-80
-100
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (rad/s


90
0.5
120
60
0.4
0.3
150
30
0.2
0.1
180
0
210
330
240
300
270

EX

a- Find analytical expressions for the magnitude and phase responses of

$$G(s) = \frac{1}{(s+2)(s+4)}$$

b- Make plots of the log-magnitude and the phase, using log-frequency in rad/s as the ordinate.

c-Make a polar plot of the frequency response.

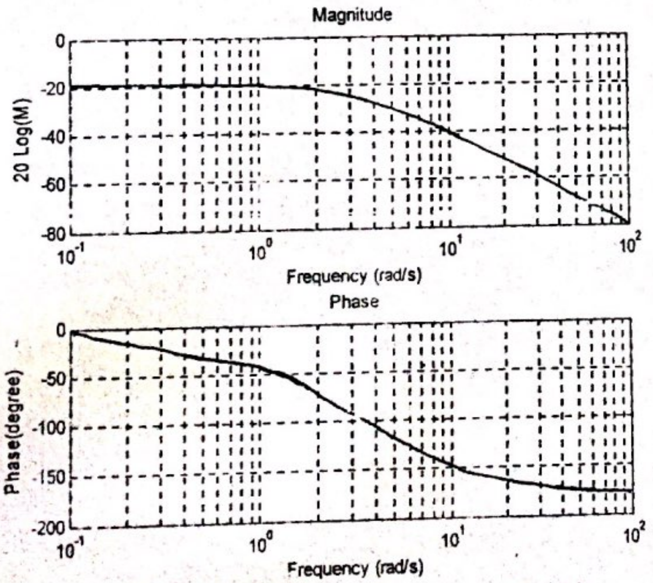
$$G(s) = \frac{1}{(s+2)(s+4)}$$

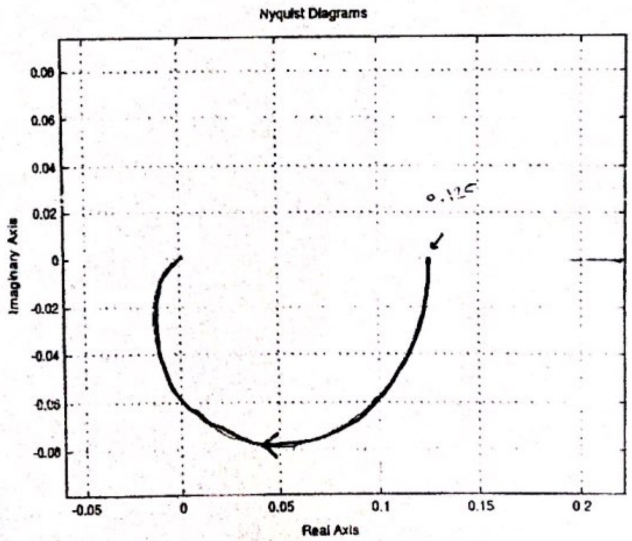
$$G(j\omega) = \frac{1}{(8-\omega^2)+j6\omega}$$

$$M(\omega) = \frac{1}{\sqrt{(8-\omega^2)^2+(6\omega)^2}}$$

$$\text{For } \omega < \sqrt{8},\ \phi(\omega) = -\tan^{-1}\left(\frac{6\omega}{8-\omega^2}\right).$$

$$\text{For } \omega > \sqrt{8},\ \phi(\omega) = -\left(\pi + \tan^{-1}\left[\frac{6\omega}{8-\omega^2}\right]\right)$$





Asymptotic Approximations: Bode Plots

The log-magnitude and phase frequency response curves as functions of log ω are called **Bode plots** or **Bode diagrams**. Sketching Bode plots can be simplified because they can be approximated as a sequence of straight lines. Straight-line approximations simplify the evaluation of the magnitude and phase frequency response.

Bode Plots for G(s) = (s + a)

Consider a function, $G(s) = (s + a)$, for which we want to sketch separate logarithmic magnitude and phase response plots. Letting $s = j\omega$, we have

$$G(j\omega) = (j\omega + a) = a\left(j\frac{\omega}{a} + 1\right)$$

At low frequencies when ω approaches zero,

$$G(j\omega) \approx a$$

The magnitude response in dB is

$$20 \log M = 20 \log a$$

where $M = |G(j\omega)|$ and is a constant. .

At high frequencies where $\omega \gg a$,

$$G(j\omega) \approx a\left(\frac{j\omega}{a}\right) = a\left(\frac{\omega}{a}\right)\angle 90^\circ = \omega\angle 90^\circ$$

The magnitude response in dB is

$$20 \log M = 20 \log a + 20 \log \frac{\omega}{a} = 20 \log \omega$$

where $a < \omega < \infty$. Notice from the middle term that the high-frequency approximation is equal to the low-frequency approximation when $\omega = a$, and increases for $\omega > a$.

If we plot dB, 20 log M, against log ω,

$$20 \log M = 20 \log \omega$$

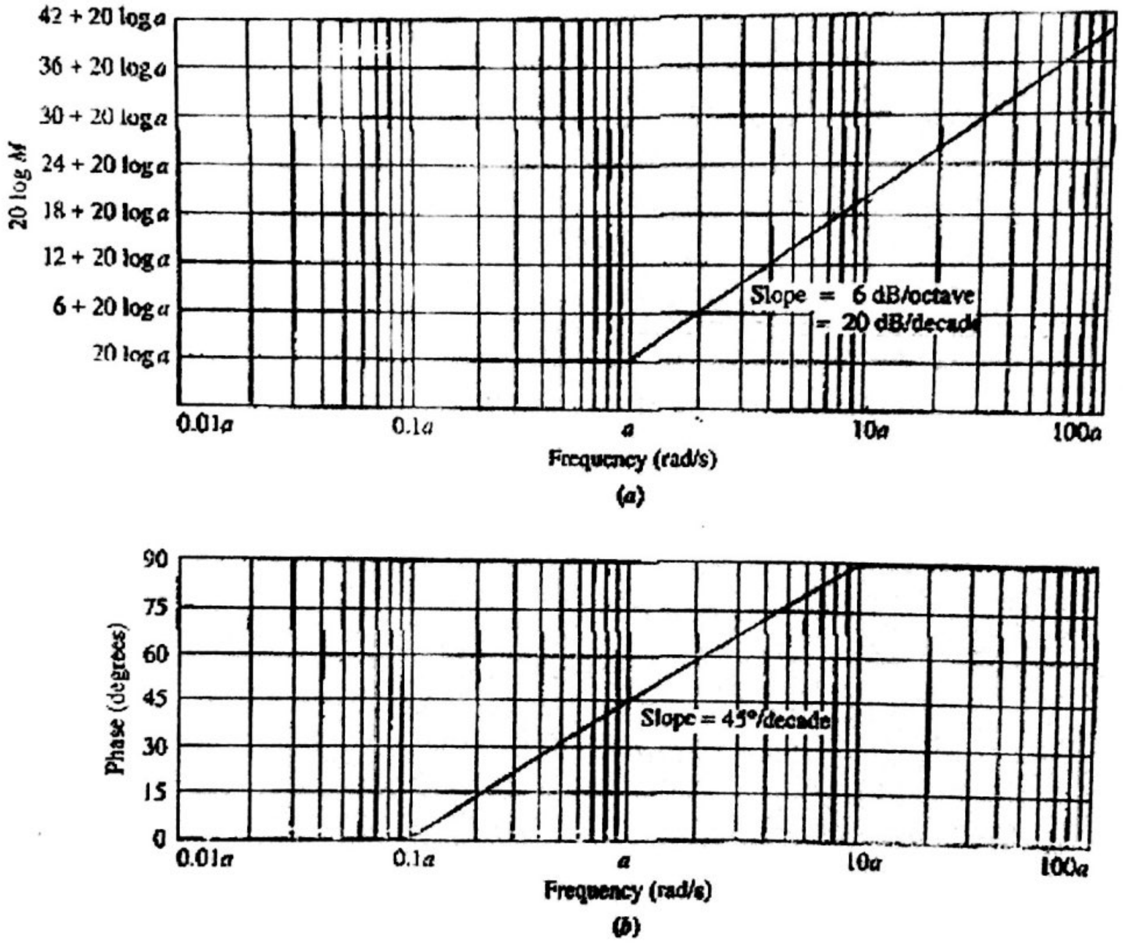
$$y = 20x$$

where $y = 20 \log M$, and $x = \log \omega$. The line has a slope of 20 when plotted as dB vs. log ω.

- Since each doubling of frequency causes 20 log ω to increase by 6 dB, the line rises at an equivalent slope of 6 dB/octave, where an *octave* is a doubling of frequency. This rise begins at $\omega = a$, where the low-frequency approximation equals the high-frequency approximation.

We call the straight-line approximations **asymptotes**. The low-frequency approximation is called the **low-frequency asymptote**, and the high-frequency approximation is called the **high-frequency asymptote**. The frequency, a, is called the **break frequency** because it is the break between the low- and the high-frequency asymptotes.

Many times it is convenient to draw the line over a decade rather than an octave, where a decade is 10 times the initial frequency. Over one decade, 20 log ω increases by 20 dB. Thus, a slope of 6 dB/octave is equivalent to a slope of 20 dB/decade.



Bode plots of $(s + a)$: **a.** magnitude plot; **b.** phase plot

Note

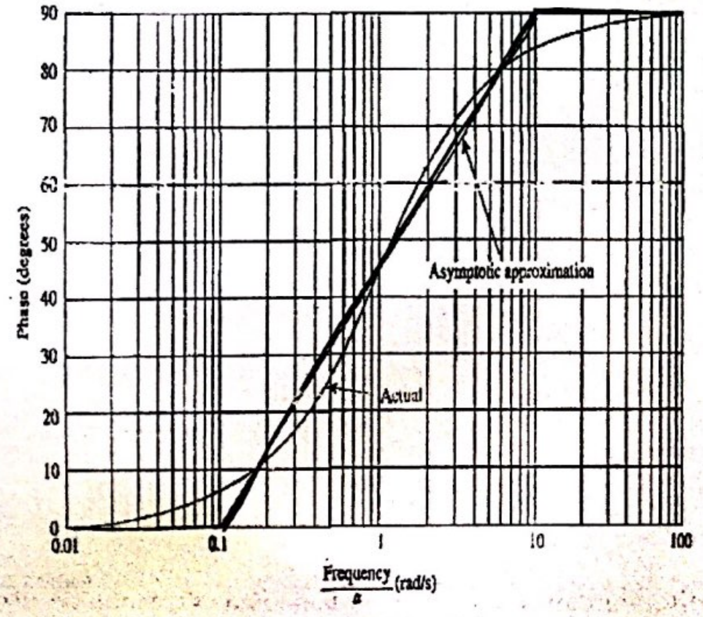
It is often convenient **to normalize** the magnitude and **scale** the frequency so that the log-magnitude plot will be 0 dB at a break frequency of unity.
To normalize (s + a), we factor out the quantity a and form a[(s/a) + 1]. The frequency is scaled by defining a new frequency variable, $s_1 = s/a$. Then the magnitude is divided by the quantity a to yield 0 dB at the break frequency. Hence, the normalized and scaled function is (s1 +1). To obtain the original frequency response, the magnitude and frequency are multiplied by the quantity a.

$G(j\omega) = j\frac{\omega}{a} + 1$

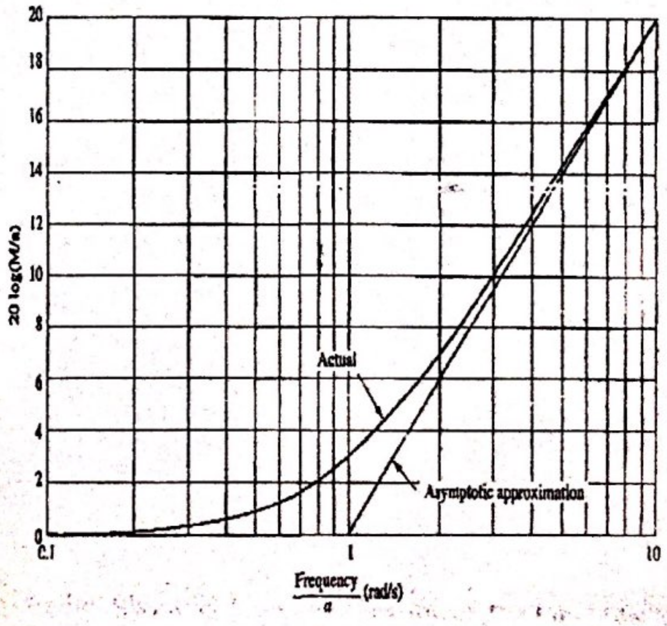
- Asymptotic: $\omega \to 0 \rightarrow G(j\omega) = 1 \Longrightarrow G_{dB} = 20 \log 1 = 0$
- Actual: $G(j\omega) = j\frac{\omega}{a} + 1 \Longrightarrow G(j\omega) = j1 + 1 \rightarrow G_{dB} = 2 \log \sqrt{2} = 3.01$ dB, $\frac{\omega}{a} = 1$, $|\sqrt{1+1}| = \sqrt{2}$

Asymptotic and actual normalized and scaled frequency response data for $(s + a)$

Frequency $\frac{\omega}{a}$ (rad/s)	$20 \log \frac{M}{a}$ (dB) Asymptotic	$20 \log \frac{M}{a}$ (dB) Actual	Phase (degrees) Asymptotic	Phase (degrees) Actual
0.01	0	0.00	0.00	0.57
0.02	0	0.00	0.00	1.15
0.04	0	0.01	0.00	2.29
0.06	0	0.02	0.00	3.43
0.08	0	0.03	0.00	4.57
0.1	0	0.04	0.00	5.71
0.2	0	0.17	13.55	11.31
0.4	0	0.64	27.09	21.80
0.6	0	1.34	35.02	30.96
0.8	0	2.15	40.64	38.66
1	0	3.01	45.00	45.00
2	6	6.99	58.55	63.43
4	12	12.30	72.09	75.96
6	15.56	15.68	80.02	80.54
8	18	18.13	85.64	82.87
10	20	20.04	90.00	84.29
20	26.02	26.03	90.00	87.14
40	32.04	32.04	90.00	88.57
60	35.56	35.56	90.00	89.05
80	38.06	38.06	90.00	89.28
100	40	40.00	90.00	89.43

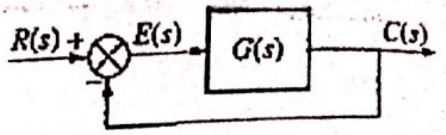


Asymptotic and actual normalized and scaled phase response of $(s + a)$



Asymptotic and actual normalized and scaled magnitude response of $(s + a)$

EX: Draw the Bode plots for the system shown in figure below, where



$$G(s) = \frac{K(s+3)}{s(s+1)(s+2)}$$

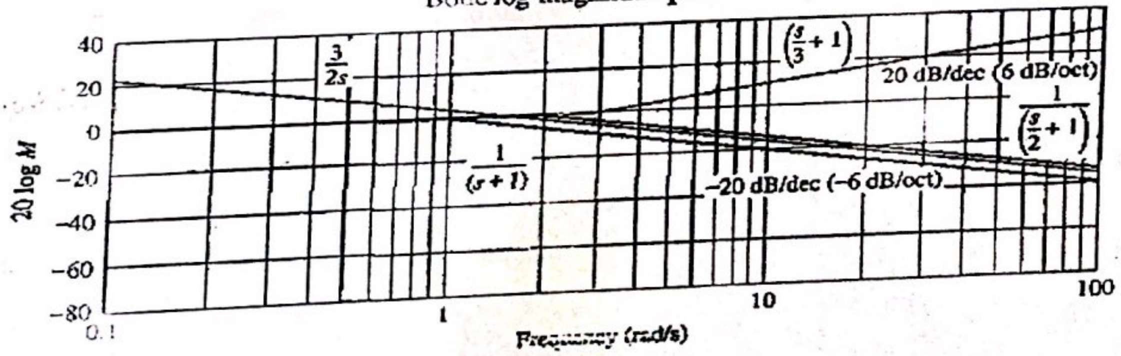
We rewrite G(s) showing each term normalized to a low-frequency gain of unity. Hence,

$$G(s) = \frac{\frac{3}{2}K(\frac{s}{3}+1)}{s(s+1)(\frac{s}{2}+1)}$$

Bode magnitude plot: slope contribution from each pole and zero

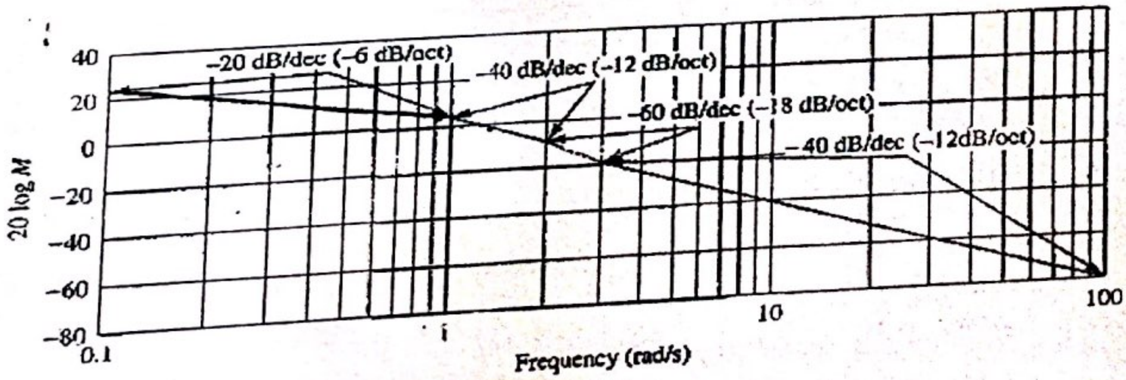
	Frequency (rad/s)			
Description	**0.1 (Start: Pole at 0)**	**1 (Start: Pole at −1)**	**2 (Start: Pole at −2)**	**3 (Start: Zero at −3)**
Pole at 0	−20	−20	−20	−20
Pole at −1	0	−20	−20	−20
Pole at −2	0	0	−20	−20
Zero at −3	0	0	0	20
Total slope (dB/dec)	−20	−40	−60	−40

Bode log-magnitude plot



(a)

a. components



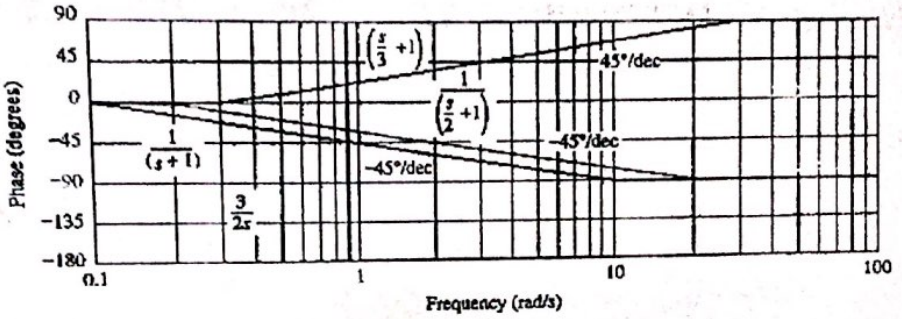
(b)

b. composite

Bode phase plot: slope contribution from each pole and zero

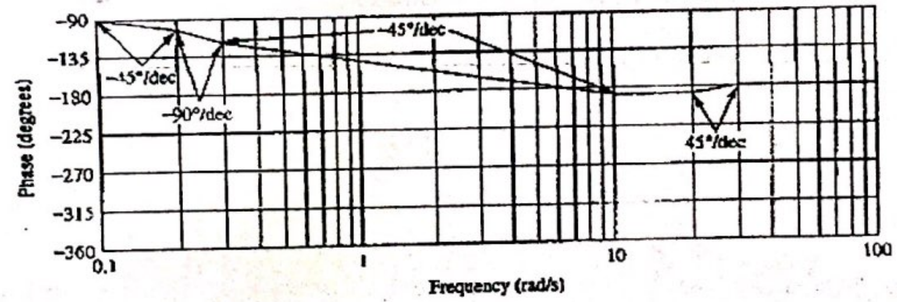
	Frequency (rad/s)					
Description	**0.1 (Start: Pole at −1)**	**0.2 (Start: Pole at −2)**	**0.3 (Start: Pole at −3)**	**10 (End: Pole at −1)**	**20 (End: Pole at −2)**	**30 (End: Zero at −3)**
Pole at −1	−45	−45	−45	0		
Pole at −2		−45	−45	−45	0	
Zero at −3			45	45	45	0
Total slope (deg/dec)	−45	−90	−45	0	45	0

a. components



(a)

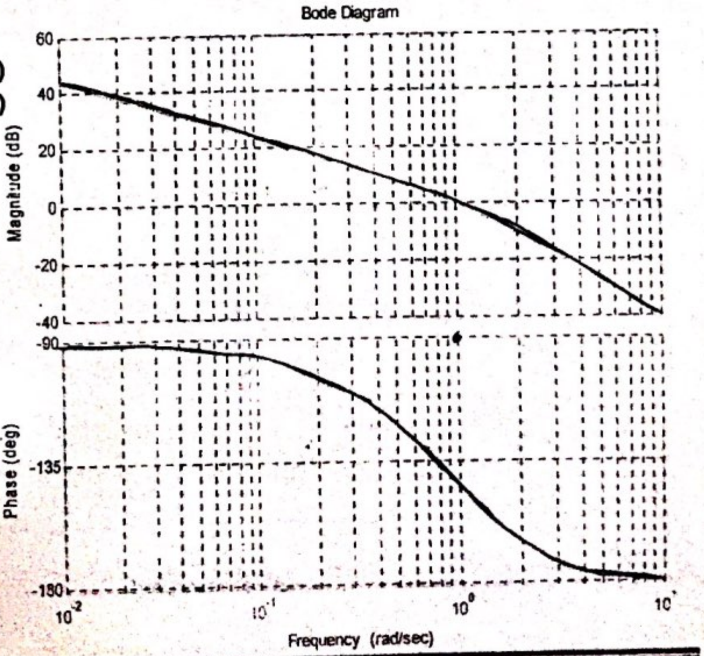
b. composite

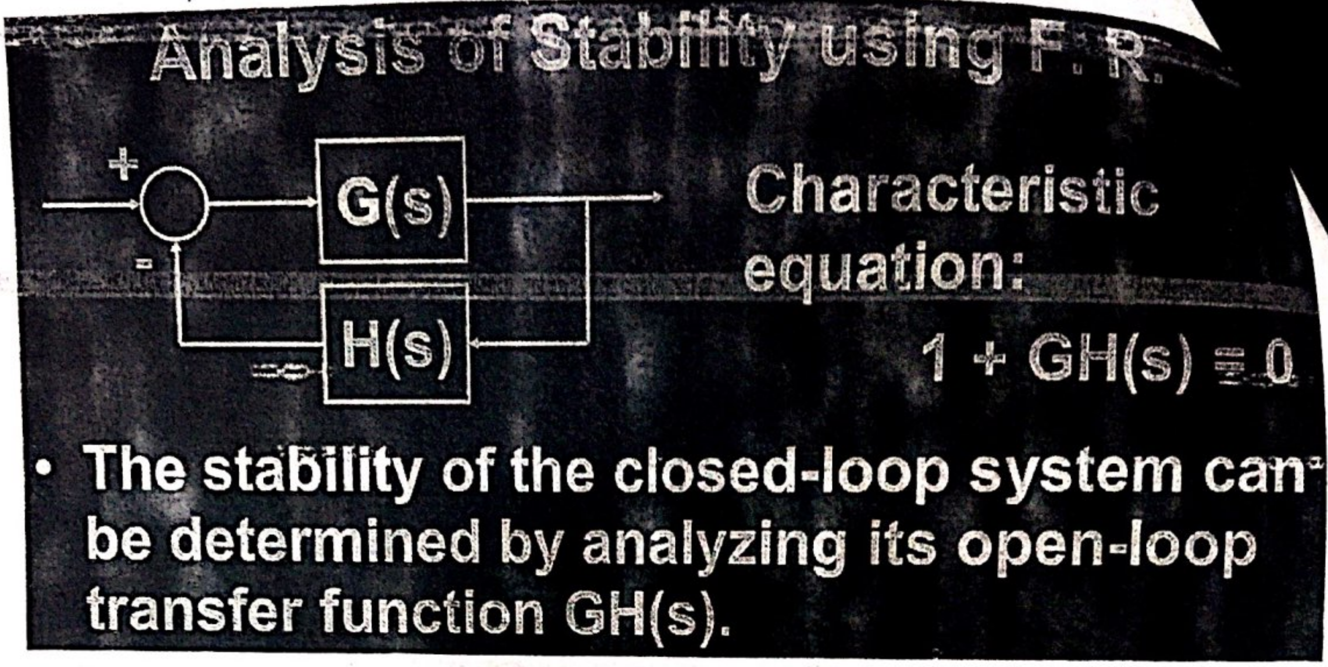


(b)

MATLAB code

```
K= input('Enter the value K:')
G= tf(K*[1 3],poly([0 -1 -2]))
bode(G)
grid
```





Analysis of Stability using F. R.

Characteristic equation:

$$1 + GH(s) = 0$$

- The stability of the closed-loop system can be determined by analyzing its open-loop transfer function GH(s).

Poles and Zeros of 1+GH(s)

$$\text{O.L.T.F.}: \; G(s)H(s) = GH(s) = \frac{\text{O.L. zeros}}{\text{O.L. poles}}$$

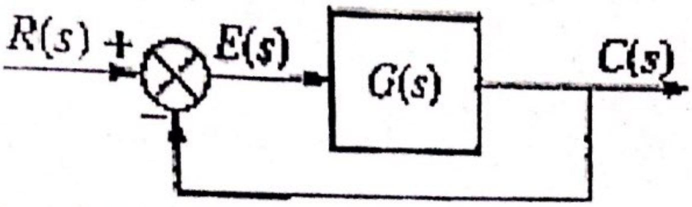
$$\text{C.L.T.F.}: \; \frac{G(s)}{1+GH(s)} = \frac{\text{C.L. zeros}}{\text{C.L. poles}}$$

$$\text{C.L.C.E.}: \; 1+GH(s) = 0$$

$$1+\frac{\text{O.L. zeros}}{\text{O.L. poles}} = 0$$

$$\frac{(\text{O.L. poles})+(\text{O.L. zeros})}{\text{O.L. poles}} = 0$$

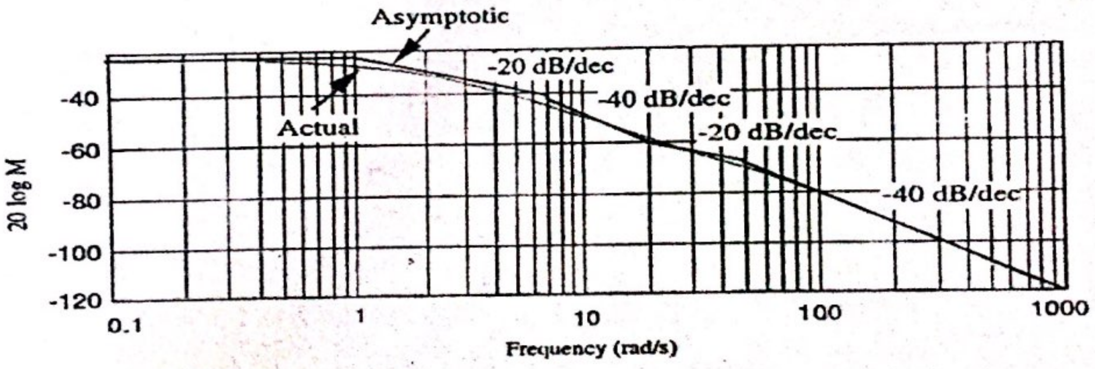
: Draw the Bode log-magnitude and phase plots for the system shown in figure below, here

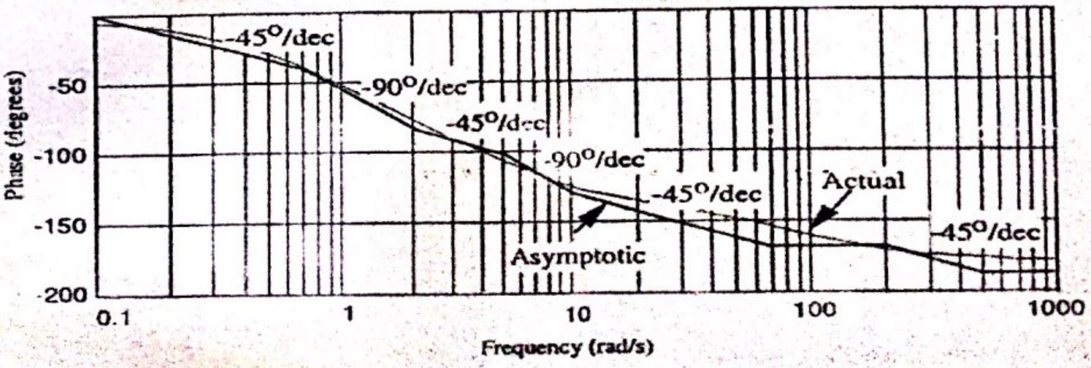


$$G(s) = \frac{(s+20)}{(s+1)(s+7)(s+50)}$$

$$G(s) = \frac{20(\frac{s}{20}+1)}{7\times 50(s+1)\left(\frac{s}{7}+1\right)(\frac{s}{50}+1)}$$

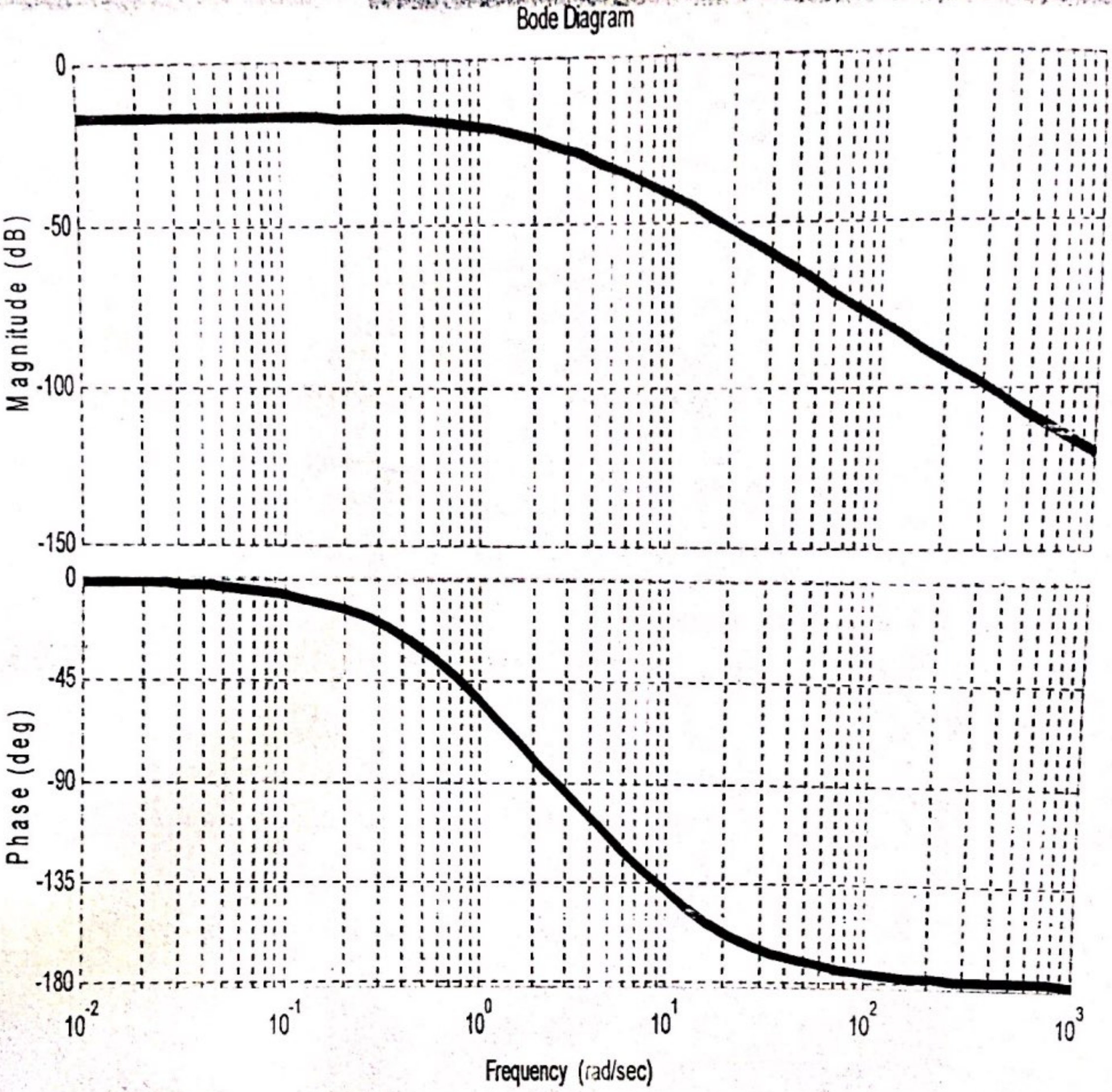
$$G(s) = \frac{(\frac{2}{35})(\frac{s}{20}+1)}{(s+1)\left(\frac{s}{7}+1\right)(\frac{s}{50}+1)}$$



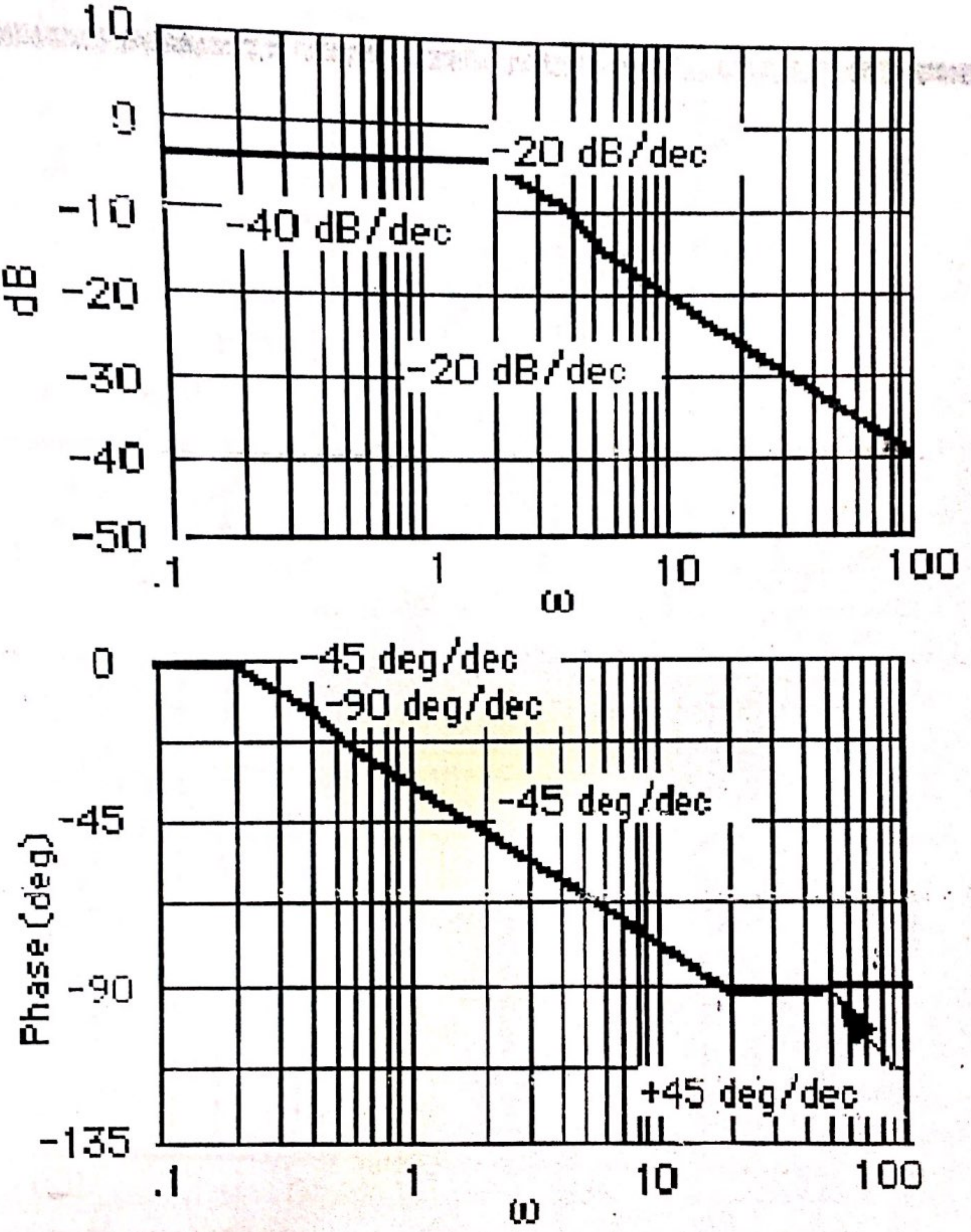


MATLAb Code

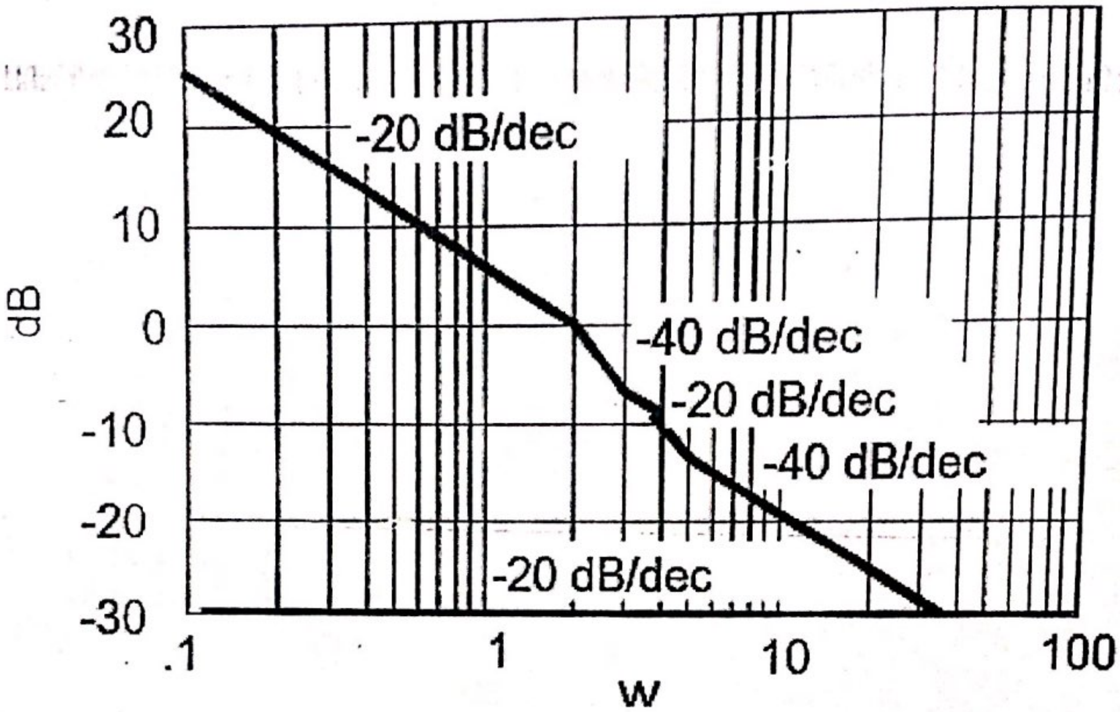
```
G= tf([1 20],poly([-1 -7 -20]))
bode(G)
grid
```

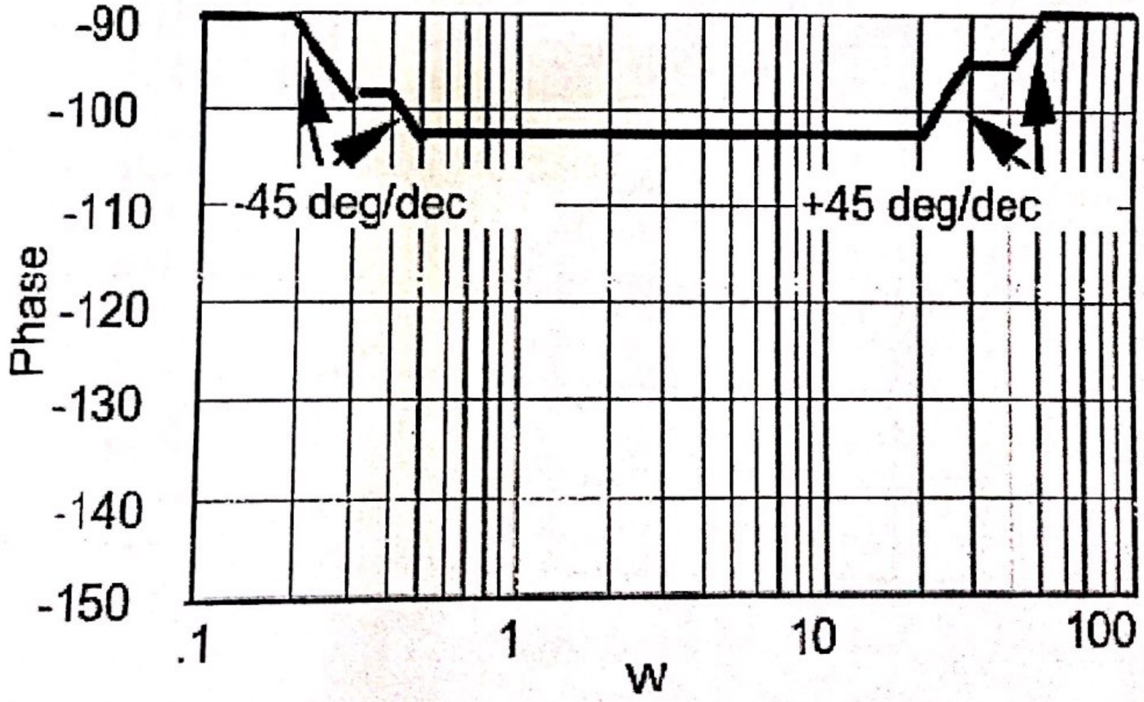


b. $G(s) = \dfrac{(s+5)}{(s+2)(s+4)}$



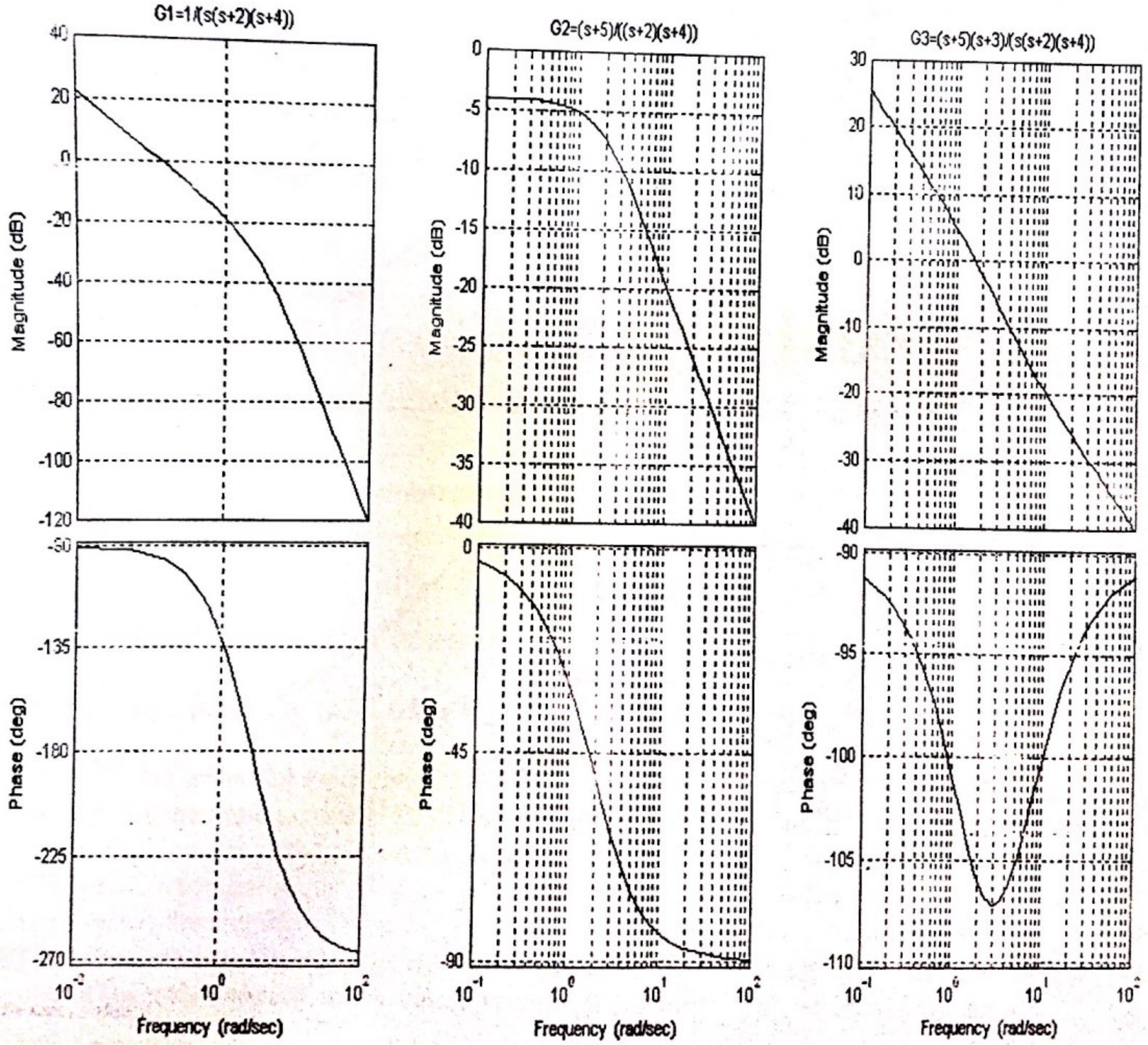
c. $G(s) = \dfrac{(s+3)(s+5)}{s(s+2)(s+4)}$





MATLAB code

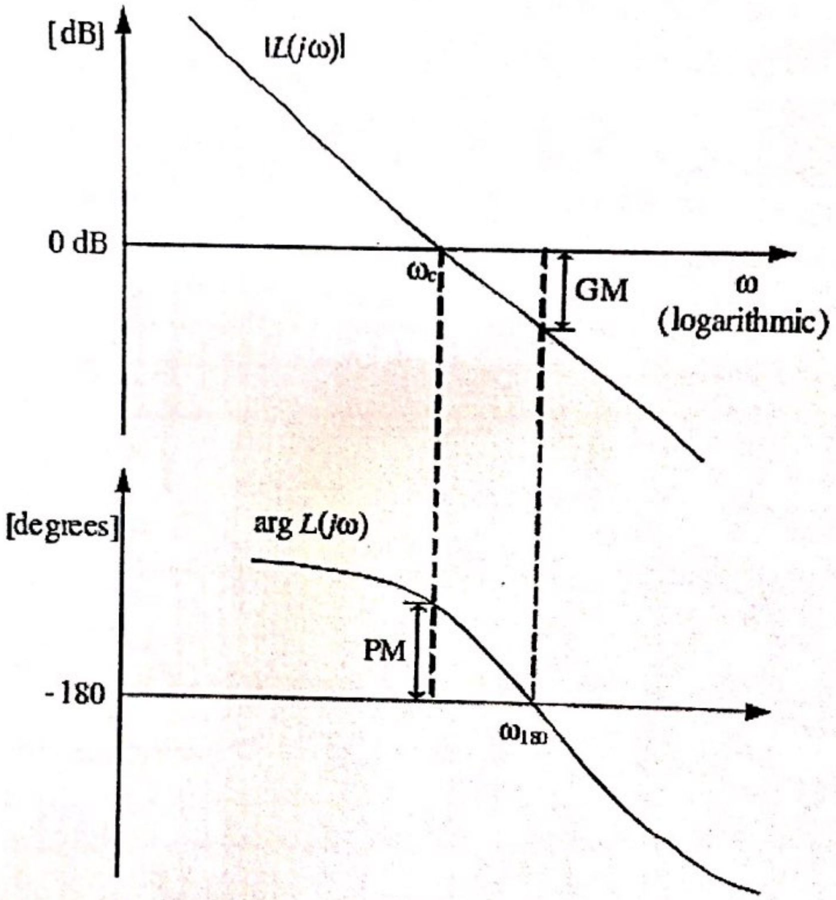
```
G1=tf(1,poly([ 0 -2 -4]));
G2=tf([1 5],poly([ -2 -4]));
G3=tf(poly([-3 -5]) ,poly([ 0 -2 -4]));
zpk(G1)
zpk(G2)
zpk(G3)

subplot(131), bode(G1); grid
title('G1=1/(s(s+2)(s+4))')
subplot(132), bode(G2); grid
title('G2=(s+5)/((s+2)(s+4))')
subplot(133), bode(G3); grid
title('G3=(s+5)(s+3)/(s(s+2)(s+4))')
```



Stability analysis in a Bode diagram

It is most common to use a Bode diagram for frequency response based stability analysis of closed loop systems. The Nyquist's Stability Criterion says: The closed loop system is marginally stable if the Nyquist curve (of L) goes through the critical point, which is the point (-1, 0).

But where is the critical point in the Bode diagram? The critical point has phase (angle) -180° and amplitude 1 = 0dB. The critical point therefore constitutes two lines in a Bode diagram: The 0dB line in the amplitude diagram and the -180 ° line in the phase diagram.



$$\boxed{GM = |L(j\omega_{180})|} \qquad PM = \angle L(j\omega_c) - (-180°)$$

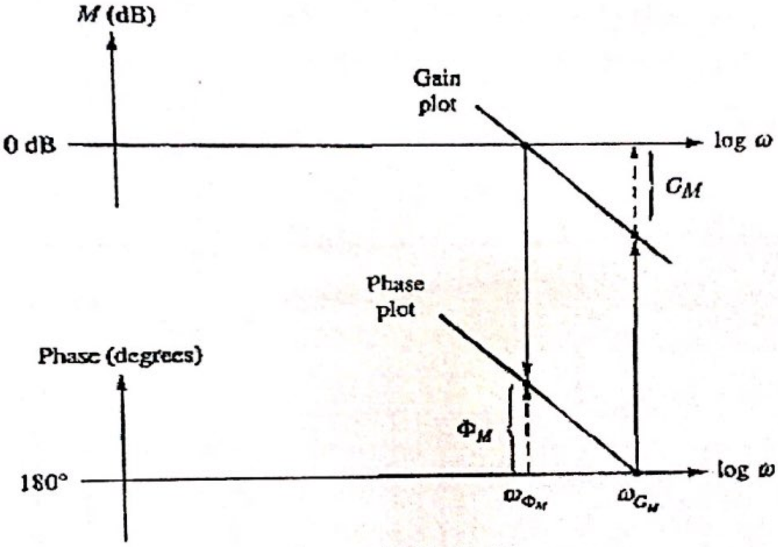
Determining Stability

We will draw a Bode log-magnitude plot and then determine the value of gain that ensures that the magnitude is less than 0 dB (unity gain) at that frequency where the phase is ± 180°.

valuating Gain and Phase Margins

Next we show how to evaluate the gain and phase margins by using Bode plots. The gain margin is found by using the phase plot to find the frequency, ω_{GM} , where the phase angle is 180°. At this frequency, we look at the magnitude plot to determine the gain margin, G_M, which is the gain required to raise the magnitude curve to 0 dB. The phase margin is found by using the magnitude curve to find the frequency, ω_{ϕ_M} where the gain is 0 dB. On the phase curve at that frequency, the phase margin is the difference between the phase value and 180°.

- **Gain Margin:** additional gain that makes the system on the verge of instability.
- **Phase Margin:** additional phase *lag* that makes the system on the verge of instability.



Gain and phase margins on the Bode diagrams

We know that the characteristic equation is

$$1 + KGH = 0$$

written for a particular gain. Then

$$KGH = -1$$

so that : **a)** when the amplitude of KGH is 1 or 0 dB and **b)** phase is 180° then we have the system just become unstable. Both the conditions have to be fulfilled. Hence, if we ask the question: How far is the system from instability ? **We get two answers**.

- The gain increment required at the frequency where phase is 180° and the phase increment required at the frequency where amplitude is 0 dB. The frequency where phase crosses 180° is called the **phase crossover** and the frequency where the amplitude crosses the 0dB point is called **the gain crossover.**

The figure below gives the bode plot for K=1000 The point in the amplitude bode plot were gain crosses **0dB** is at frequency 6.08 rad/s. At this point the phase is -157.5 degrees In order to reach 180 degree I need a phase change of 22.5 degrees (-157.5 –(-180)) which is the **Phase Margin**. The phase becomes -180 degrees at 10 rad/s and here the amplitude is less than 1 or negative in dB. I have to raise the gain by 7.96 dB before the system becomes unstable. Hence, that is my **Gain Margin**.

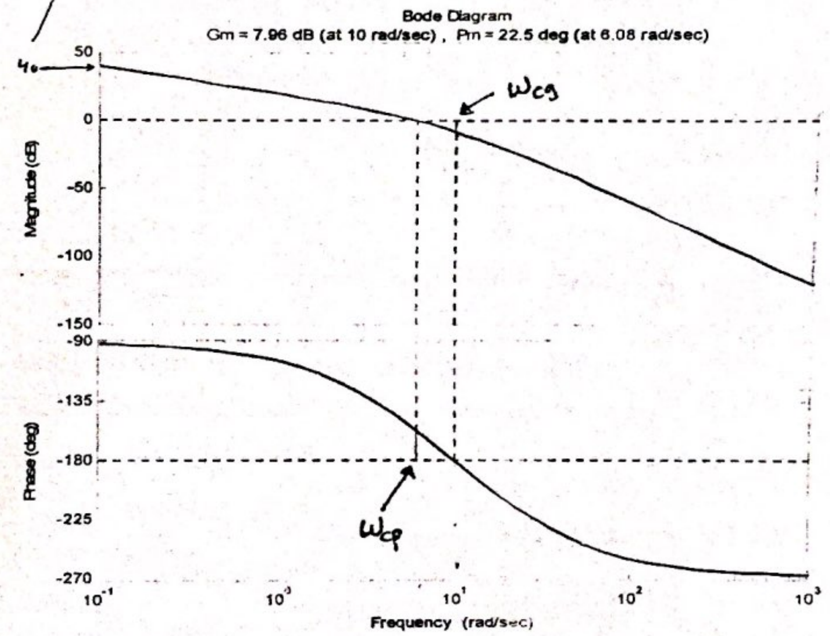
For the following transfer function find Gm and PM using MATLAB

$$G(s) = \frac{1000}{s(s+5)(s+20)}$$

```
>> g= tf(1000, poly([0 -5 -20]))

Transfer function:
         1000
--------------------
s^3 + 25 s^2 + 100 s
```

$\underline{\underline{20}}\ db + 2\log\left(\frac{1000}{100}\right) = 20 + 2 = 40\ dB$

```
>> margin(g)
```

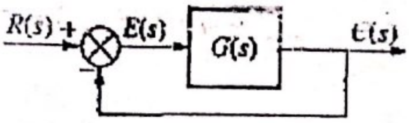


```
>> [Gm,Pm,Wcg,Wcp] = margin(g)

Gm =   2.5000        Pm =  22.5359      Wcg =   10.0000       Wcp =   6.0783


>> Gm=20*log10(Gm)

Gm =   7.9588
```

EX

For the system shown in figure below, where

$$G(s) = \frac{K}{(s+5)(s+20)(s+50)}$$



Do the following:

a. Draw the Bode log-magnitude and phase plots.

b. Find the range of K for stability from your Bode plots.

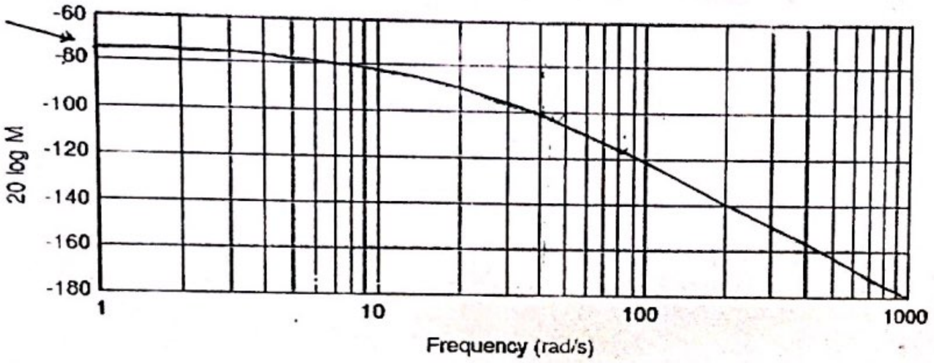
c. Evaluate gain margin, phase margin, zero dB frequency, and 180° frequency from your Bode plots for K =10000.

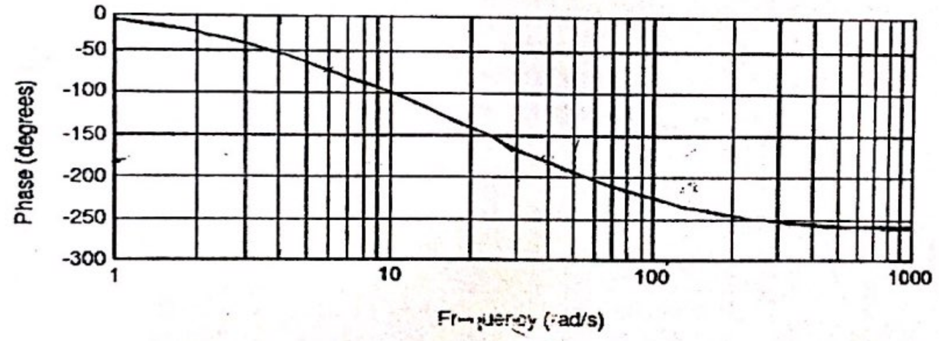
a.

$20 \log(1/5\ldots)$

$= -73.97$ dB

13-97





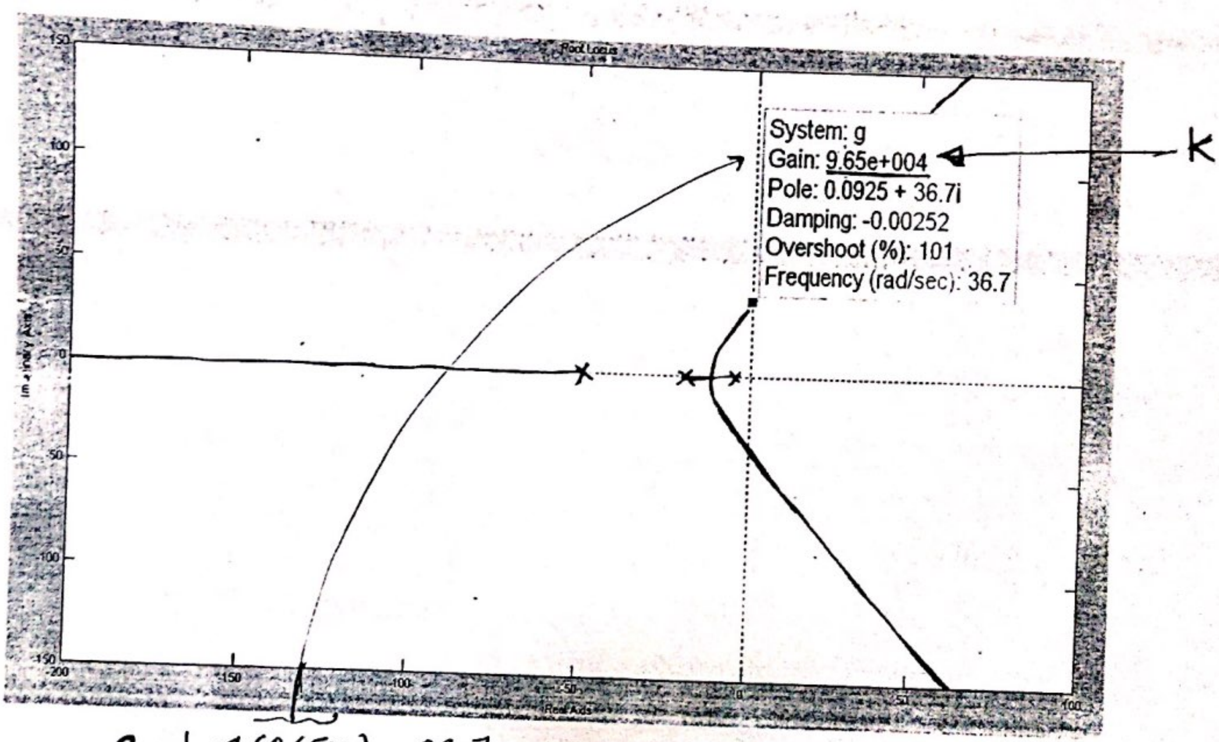
b. The phase angle is 180^0 at a frequency of 36.74 rad/s. At this frequency the gain is –99.67 dB. Therefore, $20\log K = 99.67$, or $K = 96{,}270$. We conclude that the system is stable for $K < 96{,}270$.

c. For $K = 10{,}000$, the magnitude plot is moved up $20\log 10{,}000 = 80$ dB. Therefore, the gain margin is 99.67- 80 = 19.67 dB. The 180^0 frequency is 36.7 rad/s. The gain curve crosses 0 dB at $\omega = 7.74$ rad/s, where the phase is 87.1^0. We calculate the phase margin to be $180^0 - 87.1^0 = 92.9^0$.

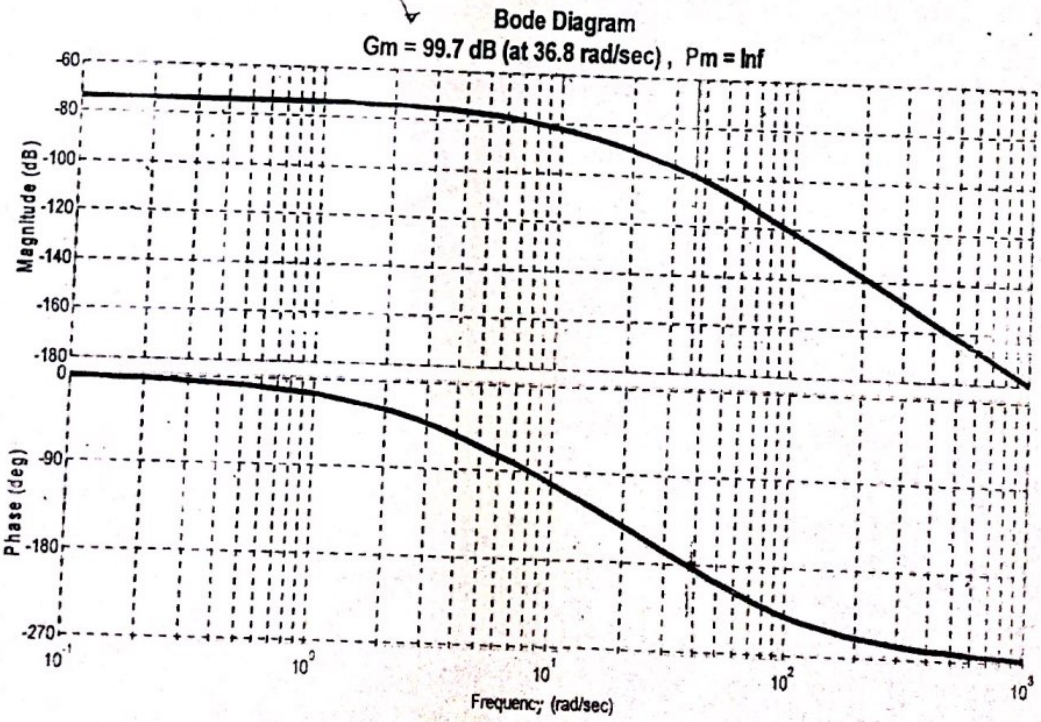
MATLAB code

```
K= input ('Enter the value of K  :')
G= tf(K,poly([-5 -20 -50])) % G(s) H(s)
rlocus(G)
figure
margin(G)
grid
[Gm ,Pm ,Wcg,Wcp]=margin(G)
Gm= 20*log10(Gm)   % gain margin db
```

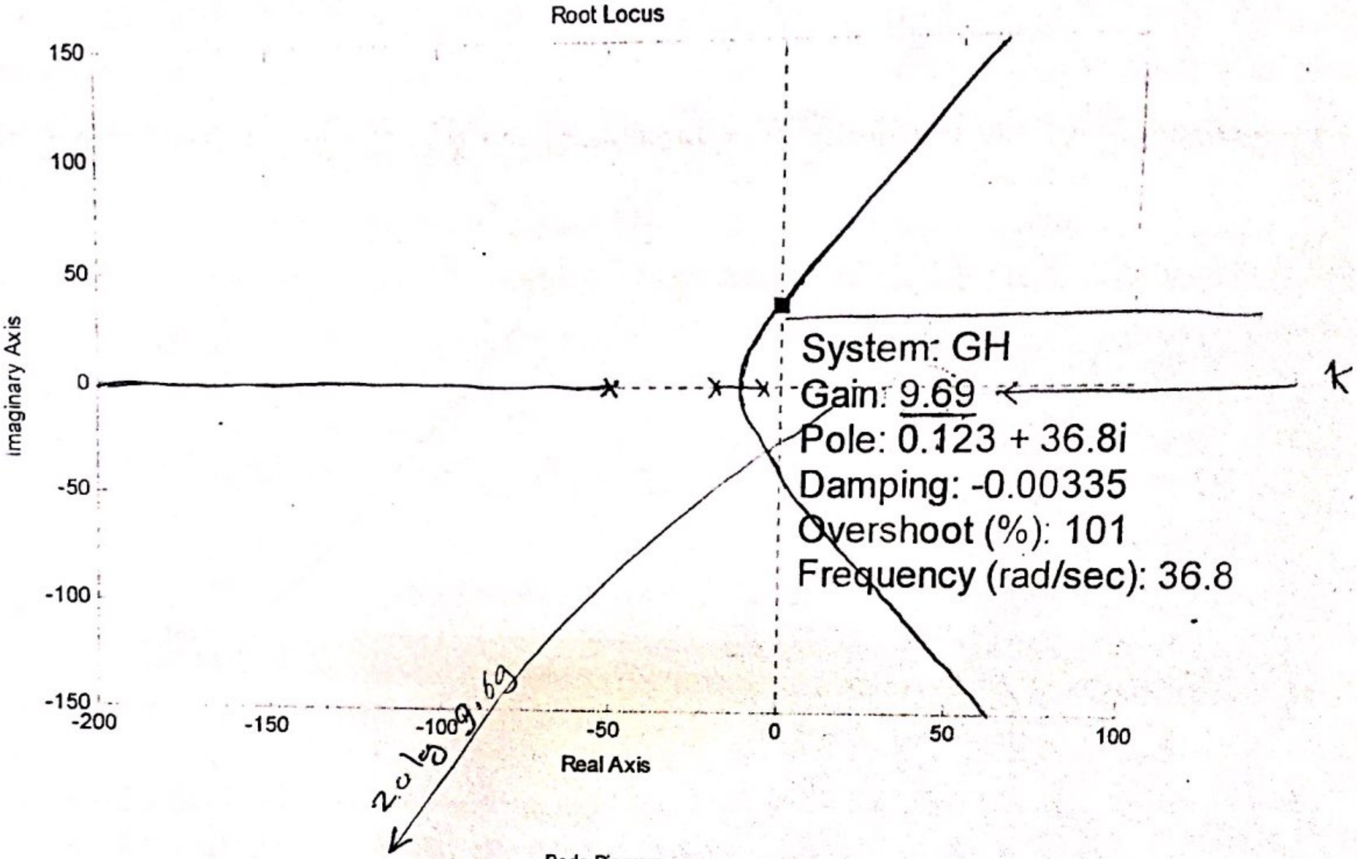
```
Enter the value of K  :1

K =
     1

Transfer function:
                1
------------------------------
s^3 + 75 s^2 + 1350 s + 5000


Gm =    9.6303e+004

Pm =     Inf

Wcg =     36.7520

Wcp =     NaN


Gm = 99.6728 db
```

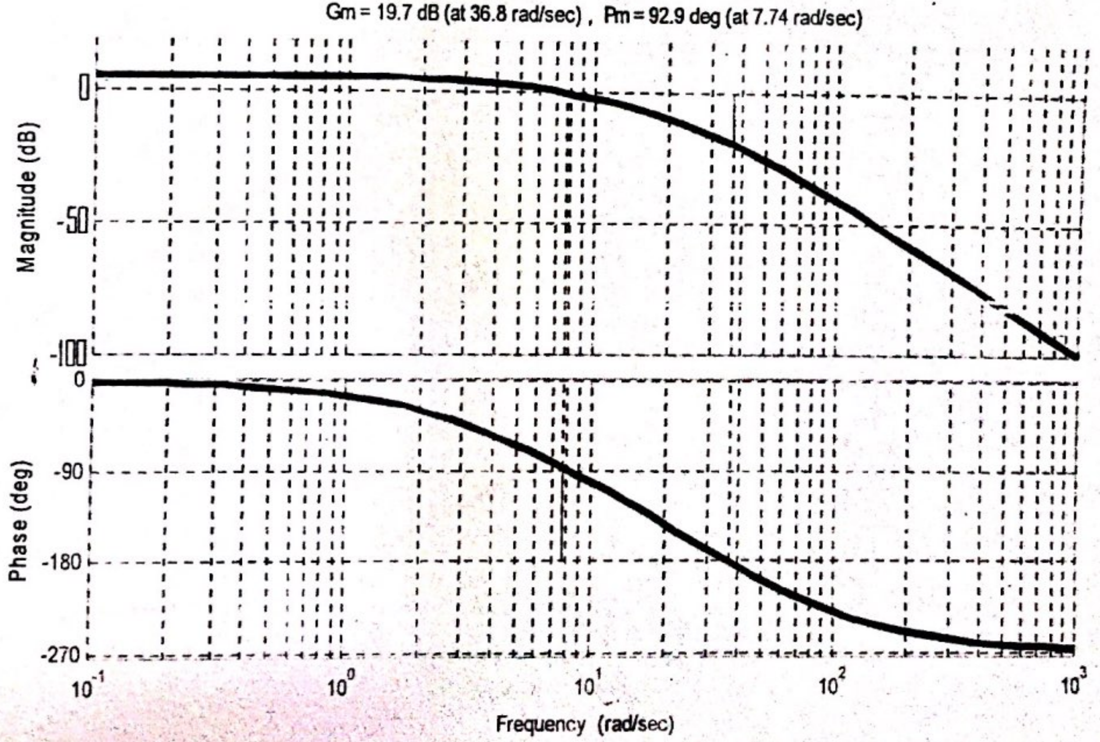


$20 \operatorname{Log}_{10}(96500) = 99.7$



For K= 10000

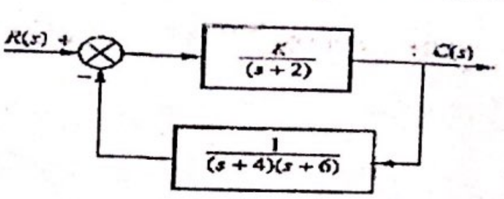

Root Locus
150
100
50
0
-50
-100
-150
Imaginary Axis
-200
-150
-100
-50
0
50
100
Real Axis
System: GH
Gain: 9.69
Pole: 0.123 + 36.8i
Damping: -0.00335
Overshoot (%): 101
Frequency (rad/sec): 36.8
K
20 log 9.69

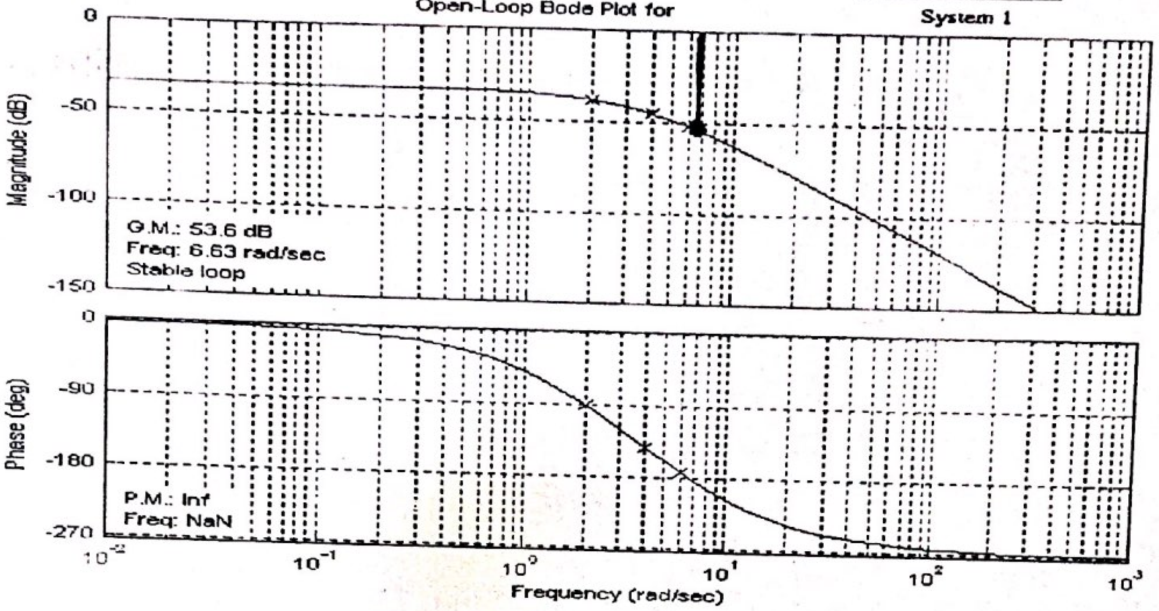

Bode Diagram
Gm = 19.7 dB (at 36.8 rad/sec) , Pm = 92.9 deg (at 7.74 rad/sec)
0
-50
-100
Magnitude (dB)
0
-90
-180
-270
Phase (deg)
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
Frequency (rad/sec)

```
                         Enter the value of K  :  10000

K =     10000


Transfer function:

                                    10000

                              -----------------------------

                          s^3 + 75 s^2 + 1350 s + 5000

 Gm =    9.6303


Pm =   92.8979


Wcg =   36.7520


Wcp =    7.7412


Gm =   19.6728 db
```

EX : Find the gain margin and phase margin if the value of K in each part the Problem is

K = 1000 , b. K = 100 and c. K = 0.1





K = 1000:

For K = 1, phase response is 180 ° at ω = 6.63 rad/s. Magnitude response is -53.6 dB at this frequency.

For K = 1000, magnitude curve is raised by 60 dB yielding + 6.4 dB at 6.63 rad/s. Thus, the gain margin is - 6.4 dB.

Phase margin: Raising the magnitude curve by 60 dB yields 0 dB at 9.07 rad/s, where the phase curve is 200.3° . Hence, the phase margin is 180° - 200.3° = = - 20.3 °

K = 100:

For K = 1, phase response is 180° at ω = 6.63 rad/s. Magnitude response is -53.6 dB at this frequency.

For K = 100, magnitude curve is raised by 40 dB yielding – 13.6 dB at 6.63 rad/s. Thus, the gain margin is 13.6 dB.

Phase margin: Raising the magnitude curve by 40 dB yields 0 dB at 2.54 rad/s, where the phase curve is 107.3°. Hence, the phase margin is 180°-107.3° = 72.7°.

K = 0.1:

For K = 1, phase response is 180° at ω = 6.63 rad/s. Magnitude response is -53.6 dB at this frequency.

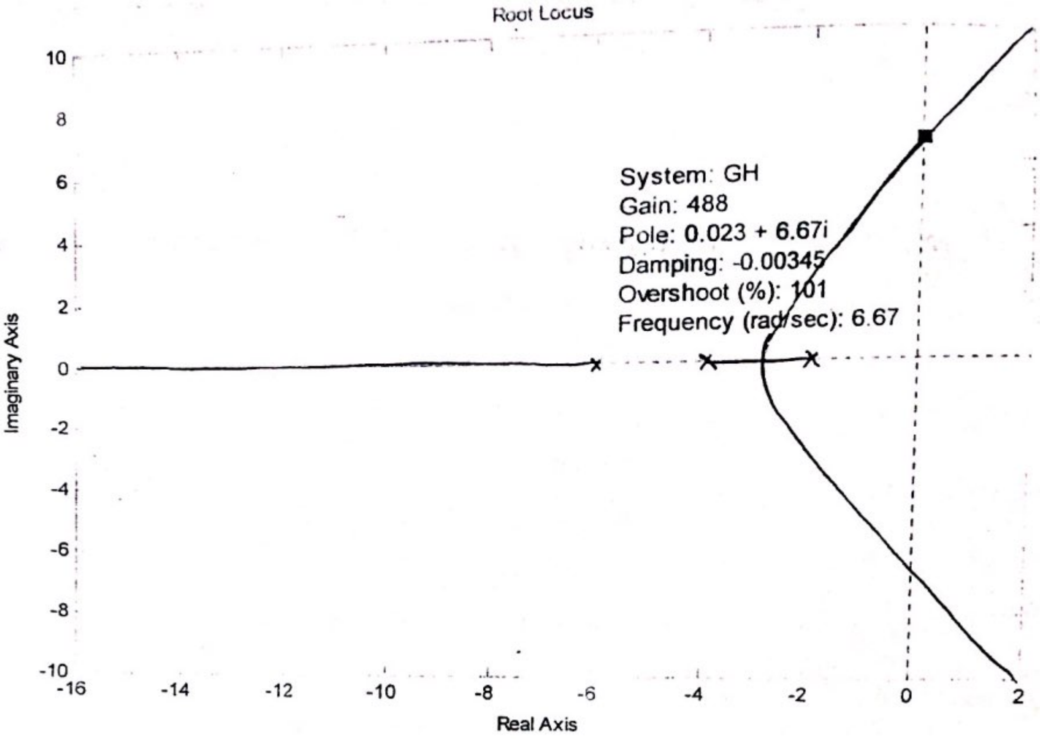
For K = 0.1, magnitude curve is lowered by 20 dB yielding – 73.6 dB at 6.63 rad/s. Thus, the gain margin is 73.6 dB..

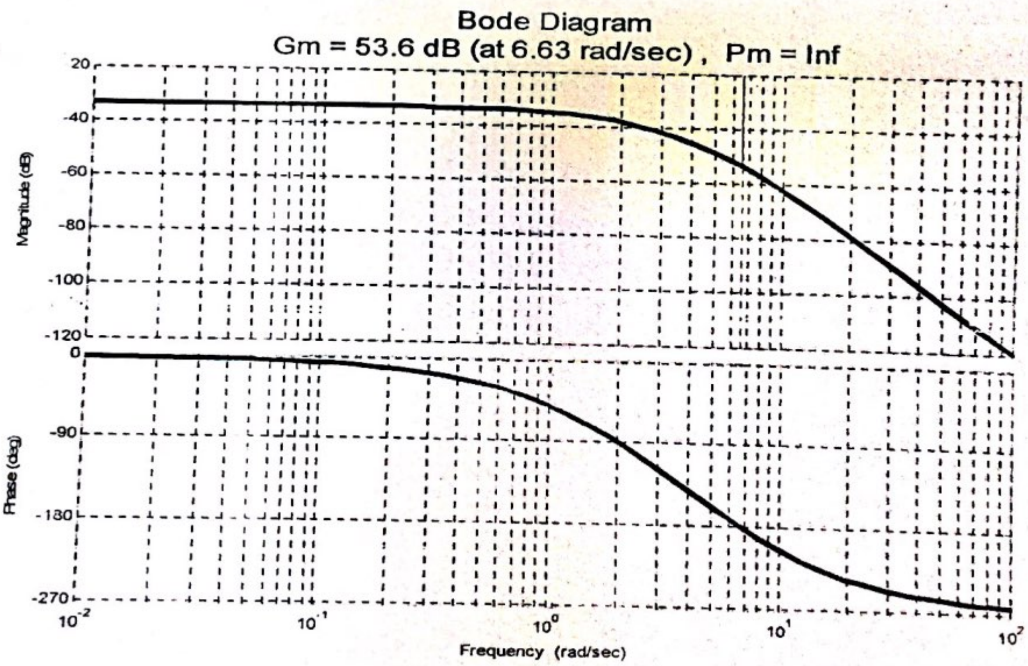
MATLAB code

```
close all
K= input ('Enter the value of K  :')
GH= tf(K,poly([-2 -4 -6])) % G(s) H(s)
rlocus(GH)
figure
margin(GH)
grid
[Gm ,Pm ,Wcg,Wcp]=margin(GH)
Gm= 20*log10(Gm)   % gain margin db
```

```
Enter the value of K  :1

K =  1


Transfer function:

        1

------------------------

s^3 + 12 s^2 + 44 s + 48


Gm =  480.0003          ✓

Pm =  Inf               ✓

Wcg =   6.6333          ✓

Wcp =   NaN             ✓

Gm =  53.6248  db
```

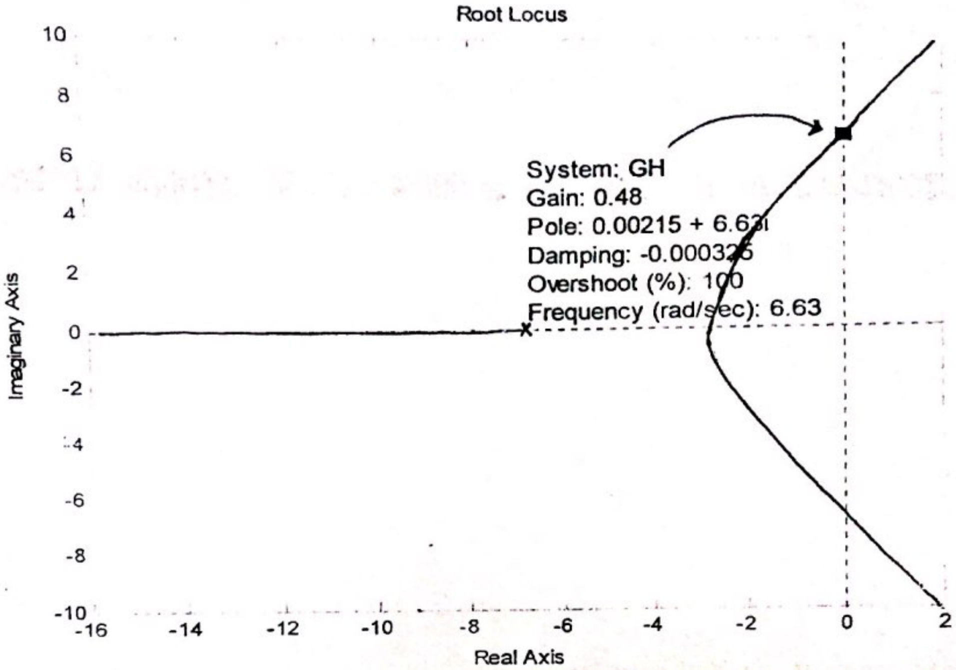
For K= 1000

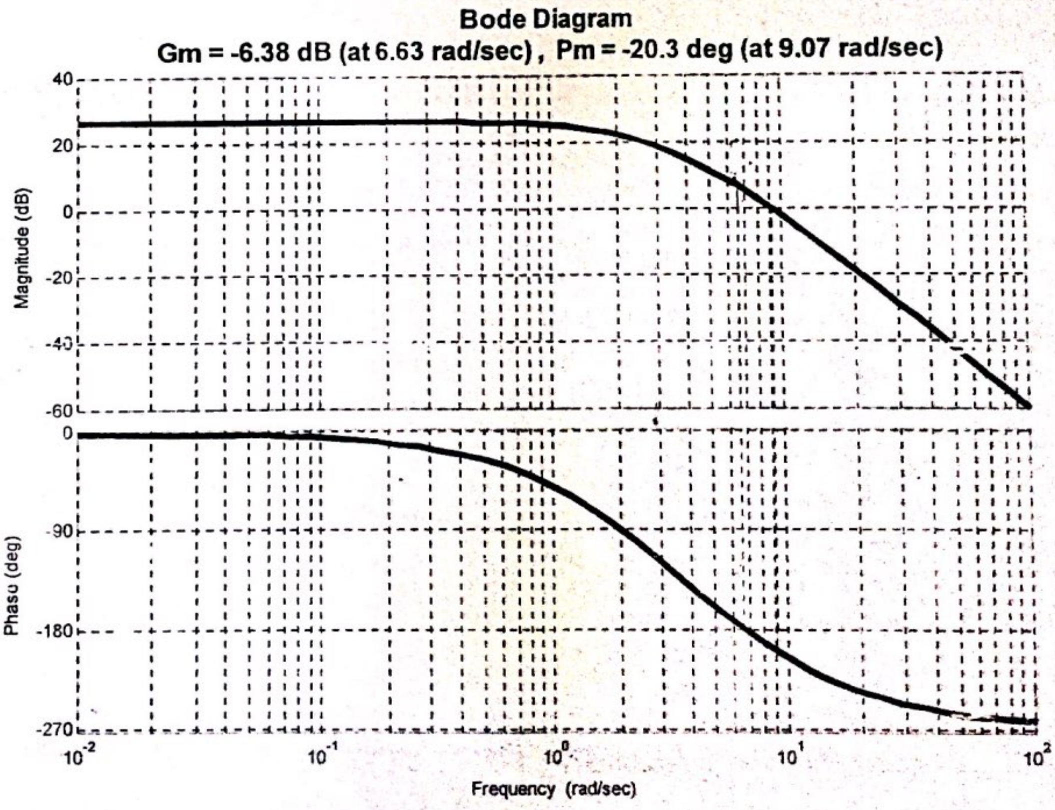




Scanned with CamScanner

For K= 1000

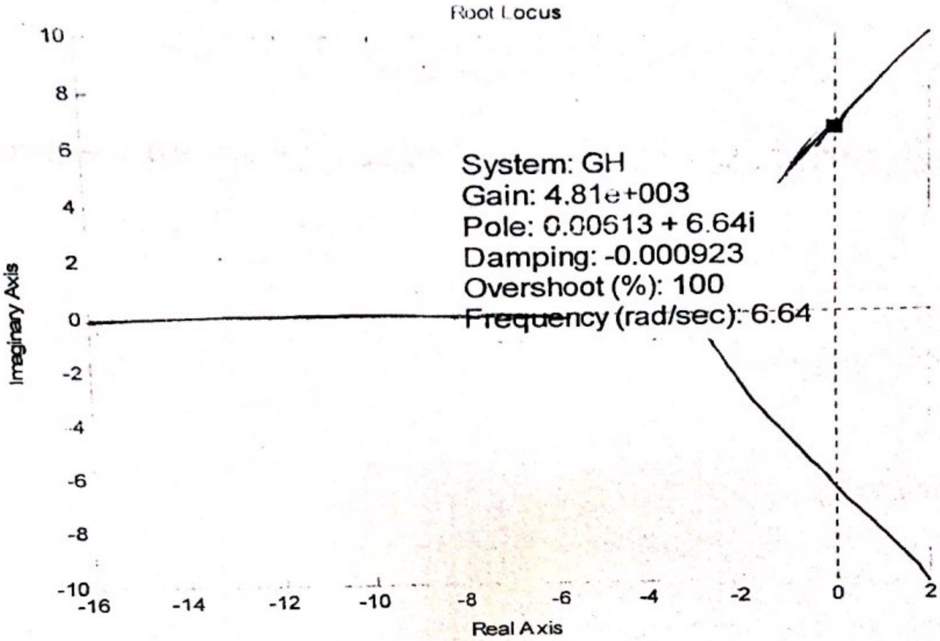

Root Locus
Imaginary Axis
Real Axis
10
8
6
4
2
0
-2
-4
-6
-8
-10
-16
-14
-12
-10
-8
-6
-4
-2
0
2
System: GH
Gain: 0.48
Pole: 0.00215 + 6.63i
Damping: -0.000325
Overshoot (%): 100
Frequency (rad/sec): 6.63

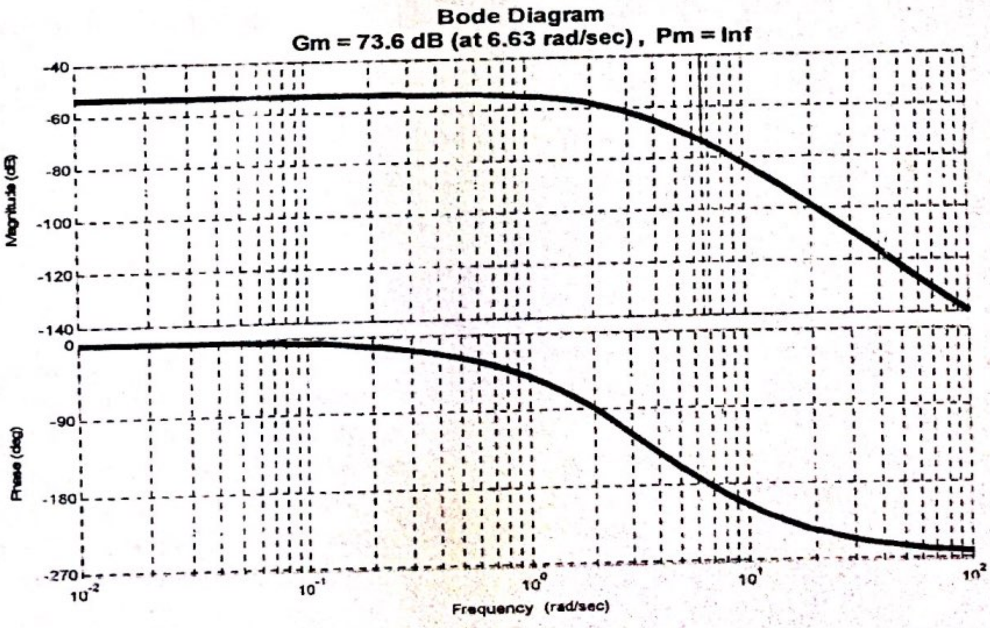

Bode Diagram
Gm = -6.38 dB (at 6.63 rad/sec) , Pm = -20.3 deg (at 9.07 rad/sec)
Magnitude (dB)
40
20
0
-20
-40
-60
Phase (deg)
0
-90
-180
-270
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (rad/sec)

For K=0.1

```
Enter the value of K :1000


K =     1000

 Transfer function:
                                        1000
                                -------------------------
                                s^3 + 12 s^2 + 44 s + 48



 Gm =    0.4800



 Pm =  -20.2512



 Wcg =    6.6333



 Wcp =    9.0654



 Gm =   -6.3752
```

For K=0.1


Root Locus
System: GH
Gain: 4.81e+003
Pole: 0.00513 + 6.64i
Damping: -0.000923
Overshoot (%): 100
Frequency (rad/sec): 6.64
Imaginary Axis
10
8
6
4
2
0
-2
-4
-6
-8
-10
-16
-14
-12
-10
-8
-6
-4
-2
0
2
Real Axis


Bode Diagram
Gm = 73.6 dB (at 6.63 rad/sec) , Pm = Inf
Magnitude (dB)
-40
-60
-80
-100
-120
-140
Phase (deg)
0
-90
-180
-270
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (rad/sec)

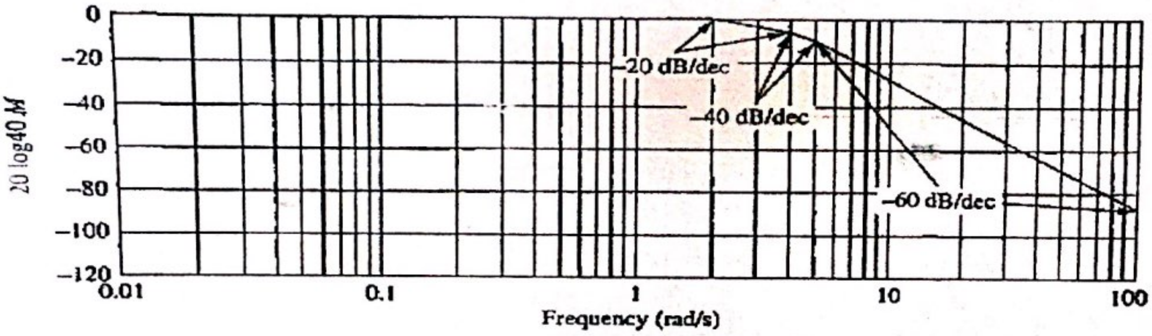
```
Enter the value of K  :0.1


K =     0.1000


Transfer function:

                        0.1
              -------------------------
              s^3 + 12 s^2 + 44 s + 48


Gm =  4.8000e+003


Pm =   Inf



Wcg =    6.6333



Wcp =   NaN



Gm =   73.6248
```

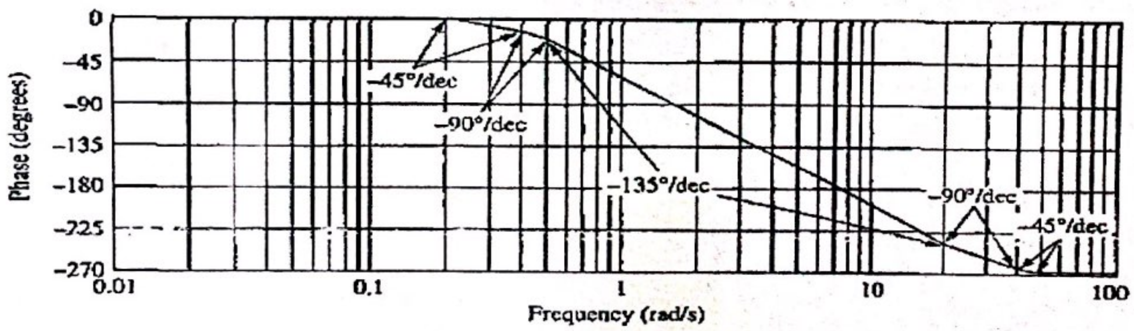
EX

Use Bode plots to determine the range of K within which the unity feedback system is stable. Let

$$G(s) = \frac{K}{(s+2)(s+4)(s+5)}$$

The low-frequency gain of G(s)H(s) is found by setting 5 to zero. Thus, the **Bode magnitude** plot starts at K/40. For convenience, let K = 40 so that the log magnitude plot starts at 0 dB. At each break frequency, 2,4, and 5, a 20 dB/decade increase in negative slope is drawn, yielding the log-magnitude plot as shown in figure below. **The phase diagram** begins at 0° until a decade below the first break frequency of 2 rad/s. At 0.2 rad/s the curve decreases at a rate of -45° /decade, decreasing an additional 45°/decade at each subsequent frequency (0.4 and 0.5 rad/s) a decade below each break. At a decade above each break frequency, the slopes are reduced by 45°/decade at each frequency.

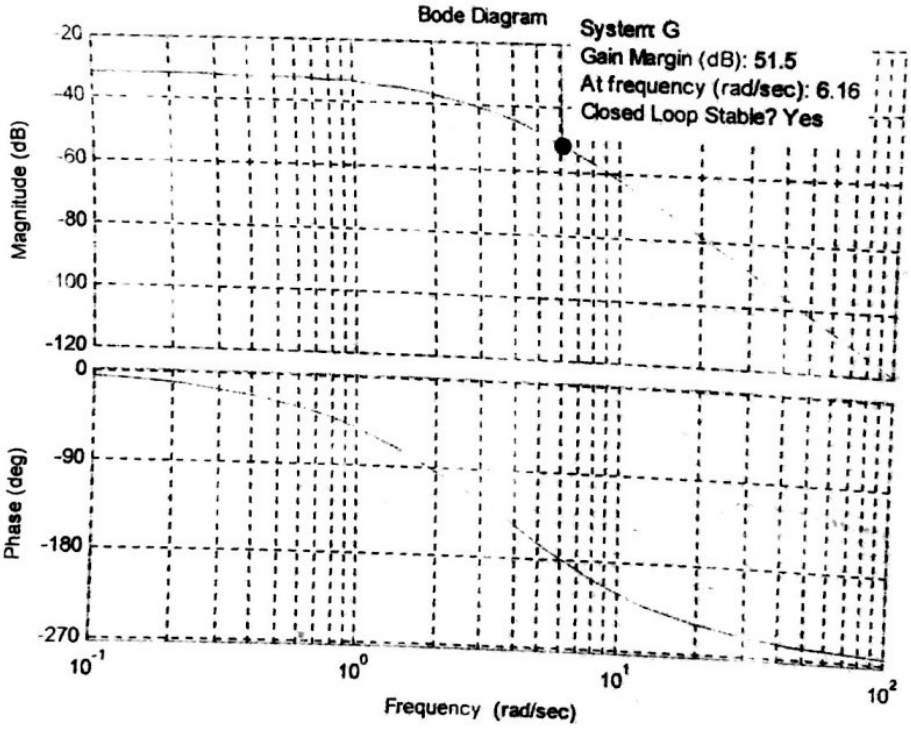




From the Bode log-magnitude plot must be less than unity when the Bode phase plot is 180°. Accordingly, we see that at a frequency of 7 rad/s, when the phase plot is -180°, the magnitude plot is -20 dB. Therefore, an increase in gain of +20 dB is possible before the system becomes unstable. Since the gain plot was scaled for a gain of 40, +20 dB (a gain of 10) represents the required increase in gain above 40. Hence, the gain for instability is 40 x 10 = 400. The final result is 0 < K < 400 for stability.

MATLAb code

```
numg=1;                                  % Define numerator of G (s
deng=poly ([-2 -4 -5]) ;                 % Define denominator of G (s
'G(s)'
G=tf(numg,deng)                          % Create and display G (s)
bode(G)
grid
[Gm,Pm,Wcg,Wcp]=margin (G) ;  % Find margins and margin frequencies.
margins= [20*log10(Gm) , Pm , Wcg ,Wcp]
K=Gm                                      % Display K for stability.
```



```
margins =

   51.5499        Inf     6.1644        NaN


K =

  378.0020
```

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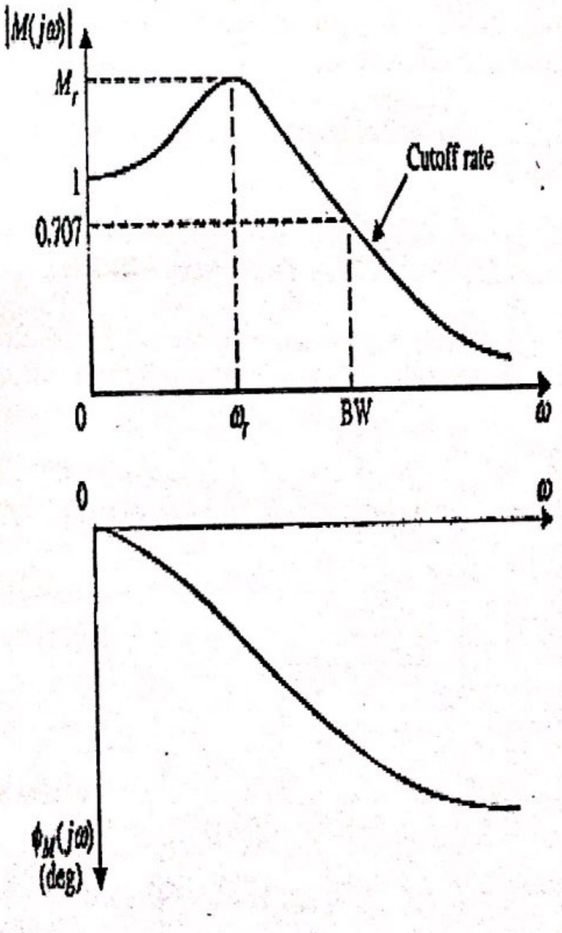
NYQUIST STABILITY CRITERION: FUNDAMENTALS

Thus far we have presented two methods of determining the stability of linear SISO systems: the **Routh-Hurwitz criterion** and the **root-locus method** of determining stability by locating the roots of the characteristic equation in the s-plane. Of course, if the coefficients of the characteristic equation are all known, we can solve for the roots of the equation by use of MATLAB.

Frequency-Domain Specifications

1- Resonant Peak Mr The resonant peak Mr is the maximum value of $|M(jw)|$. In general, the magnitude of Mr gives indication on the relative stability of a stable closed-loop system. Normally, a large Mr corresponds to a large maximum overshoot of thestep response. For most control systems, it is generally accepted in practice that the desirable value of Mr should be between 1.1 and 1.5.

2- The resonant frequency wr, is the frequency at which the peak resonance Mr occurs.



Typical gain-phase characteristics of a feedback control syster

3- Bandwidth BW
The bandwidth BW is the frequency at which $|M(jw)|$ drops to 70.7 % of, or 3 dB down from, its zero-frequency value. In general, the bandwidth of a control system gives indication on the transient-response properties in the time domain. A large bandwidth corresponds to a faster rise time, since higher-frequency signals are more easily passed through the system. Conversely, if the bandwidth is small, only signals of relatively low frequencies are passed, and the time response will be slow and sluggish. Bandwidth also indicates the noise-filtering characteristics and the robustness of the system. The robustness represents a measure of the sensitivity of a system to parameter variations. A robust system is one that is insensitive to parameter variations.

The **Nyquist criterion** is a semigraphical method that determines the stability of a closed loop system by investigating the properties of the frequency-domain plot, the Nyquist plot, of the loop transfer function G(s)H(s), or L(s). Specifically, the Nyquist plot of L(s) is a plot of L(jw) in the polar coordinates of Im[L(jw)] versus Re[L(jw)] as w varies from 0 to infinity.

This is another example of using the properties of the loop transfer function to find the performance of the closed-loop system. The Nyquist criterion has the following features that make it an alternative method that is attractive for the analysis and design of control systems.

1- In addition to providing the absolute stability, like the Routh-Hurwitz criterion, the Nyquist criterion also gives information on the relative stability of a stable system and the degree of instability of an unstable system. It also gives an indication of how the system stability may be improved, if needed.

2. The Nyquist plot of G(s)H(s) or of L(s) is very easy to obtain, especially with the aid of a computer.

3. The Nyquist plot of G(s)H(s) gives information on the frequency-domain characteristics such as Mr ,wr , BW, and others with ease.

4. The Nyquist plot is useful for systems with pure time delay that cannot be treated with the Routh-Hurwitz criterion and are difficult to analyze with the root-locus method.

EX

Sketch the bode plot and Nyquist diagram for following system with unity feedback.

$$G(s) = \frac{1}{(s+2)}$$

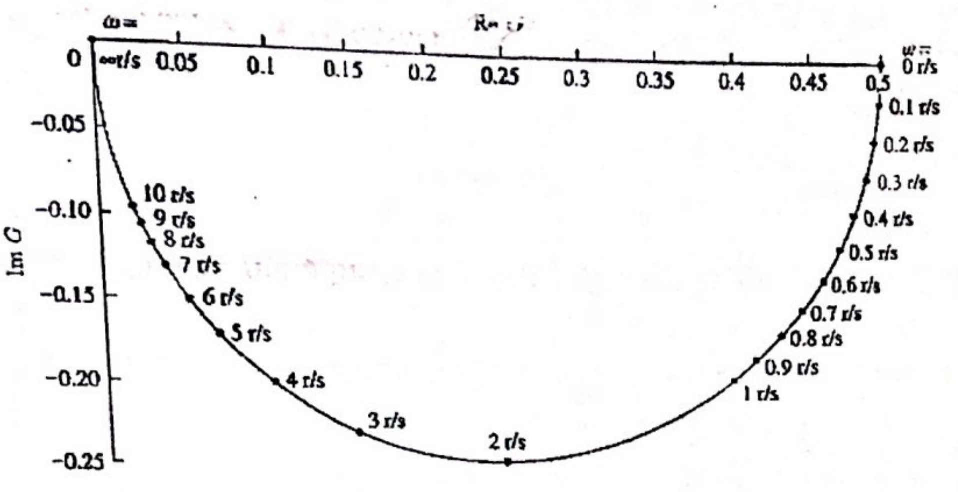
$$s = j\omega$$

$$G(j\omega) = \frac{1}{j\omega + 2}$$

$$G(j\omega) = \frac{1}{2+j\omega} \times \frac{2-j\omega}{2-j\omega} = \frac{2-j\omega}{4+\omega^2} = \frac{2}{4+\omega^2} - j\frac{\omega}{4+\omega^2}$$

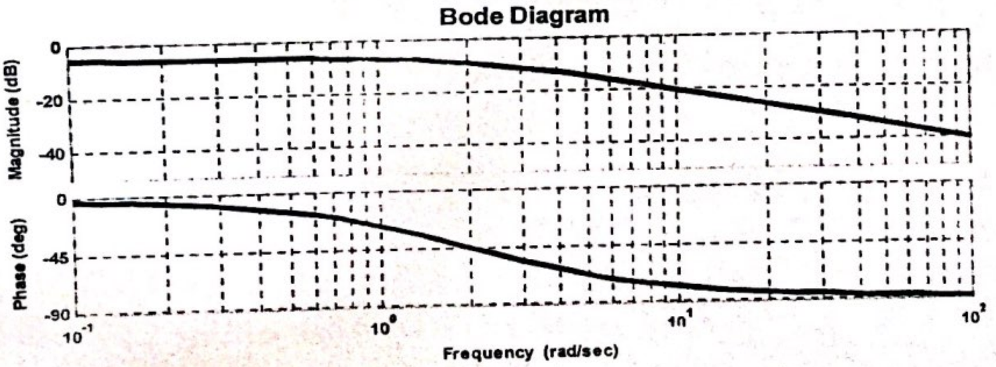
Magnitude frequency response $|G(j\omega)| = M(\omega) = \frac{1}{\sqrt{\omega^2+4}}$

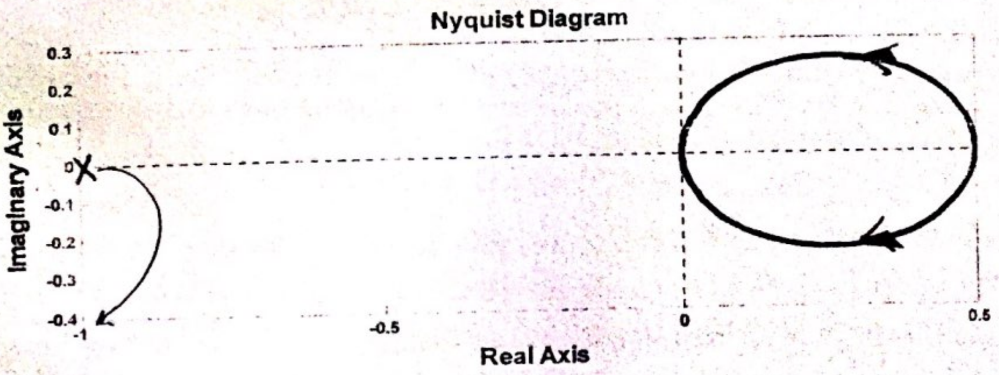
Phase frequency response $\angle G(j\omega) = \emptyset(\omega) = -\tan^{-1}\left(\frac{\omega}{2}\right)$



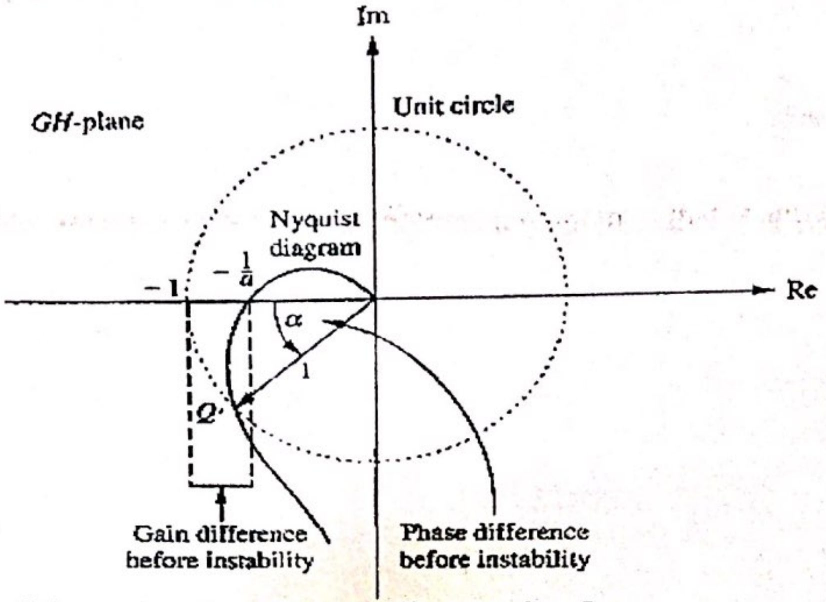
Note: r/s = rad/s

```
G= tf(1,[1 2])
subplot(211);bode(G)
subplot(212);nyquist(G)
```





Margin and Phase Margin via the Nyquist Diagram


Im
GH-plane
Unit circle
Nyquist diagram
$-\frac{1}{a}$
-1
Re
α
1
Q
Gain difference before instability
Phase difference before instability

Gain margin = $G_M = 20 \log a$ Phase margin = $\Phi_M = \alpha$

Nyquist diagram showing gain and phase margins

EX

Find the gain and phase margin for the following system if K=6, with unity feedback.

$$G(s)H(s) = \frac{k}{(s^2+2s+2)(s+2)}$$

$$G(j\omega)H(j\omega) = \left.\frac{6}{(s^2+2s+2)(s+2)}\right|_{s\to j\omega}$$

$$\mathrm{Re}\left[G(\sqrt{6})\right] = \frac{6\times4(1-6)^2}{16(1-6)^2+6(6-6)^2} \doteq -0.3$$

$$= \frac{6[4(1-\omega^2) - j\omega(6-\omega^2)]}{16(1-\omega^2)^2 + \omega^2(6-\omega^2)^2}$$

The Nyquist diagram crosses the real axis at a frequency of $\sqrt{6}$ rad/s. The real part is calculated to be -0.3. Thus, the gain can be increased by $(1/0.3) = 3.33$ before the real part becomes -1. Hence, the gain margin is

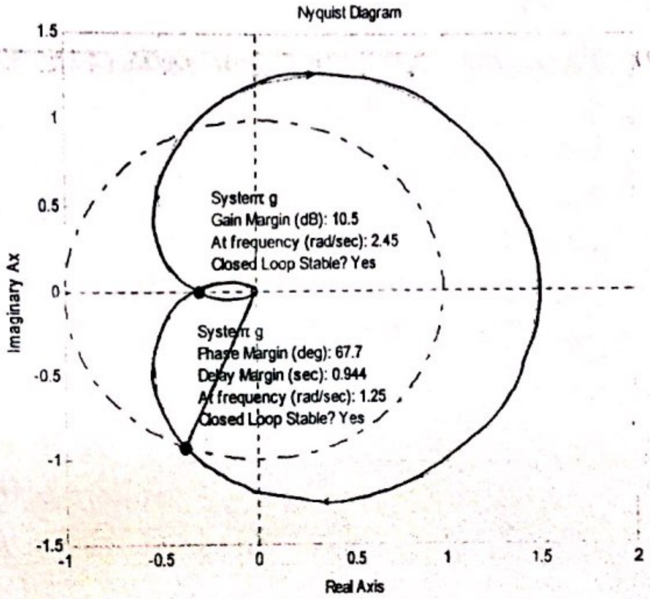
$$G_M = 20 \log 3.33 = 10.45 \text{ dB}$$

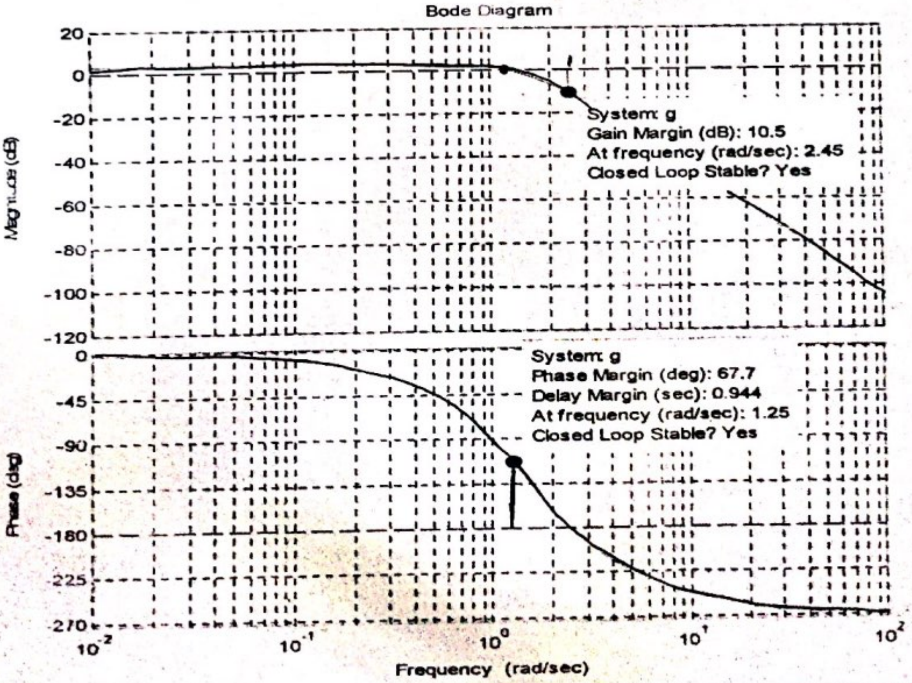
using Bode plots. $G(j\omega)H(j\omega)$ has unity gain at a frequency of 1.253 rad/s. At this frequency, the phase angle is $-112.3°$ The difference between this angle and $-180°$ is $67.7°$, which is the phase margin.

```
>> g= tf(6,poly([roots([1 2 2])' -2]))

Transfer function:
          6
---------------------
s^3 + 4 s^2 + 6 s + 4

>> bode(g)
>> figure
>> nyquist(g)
```

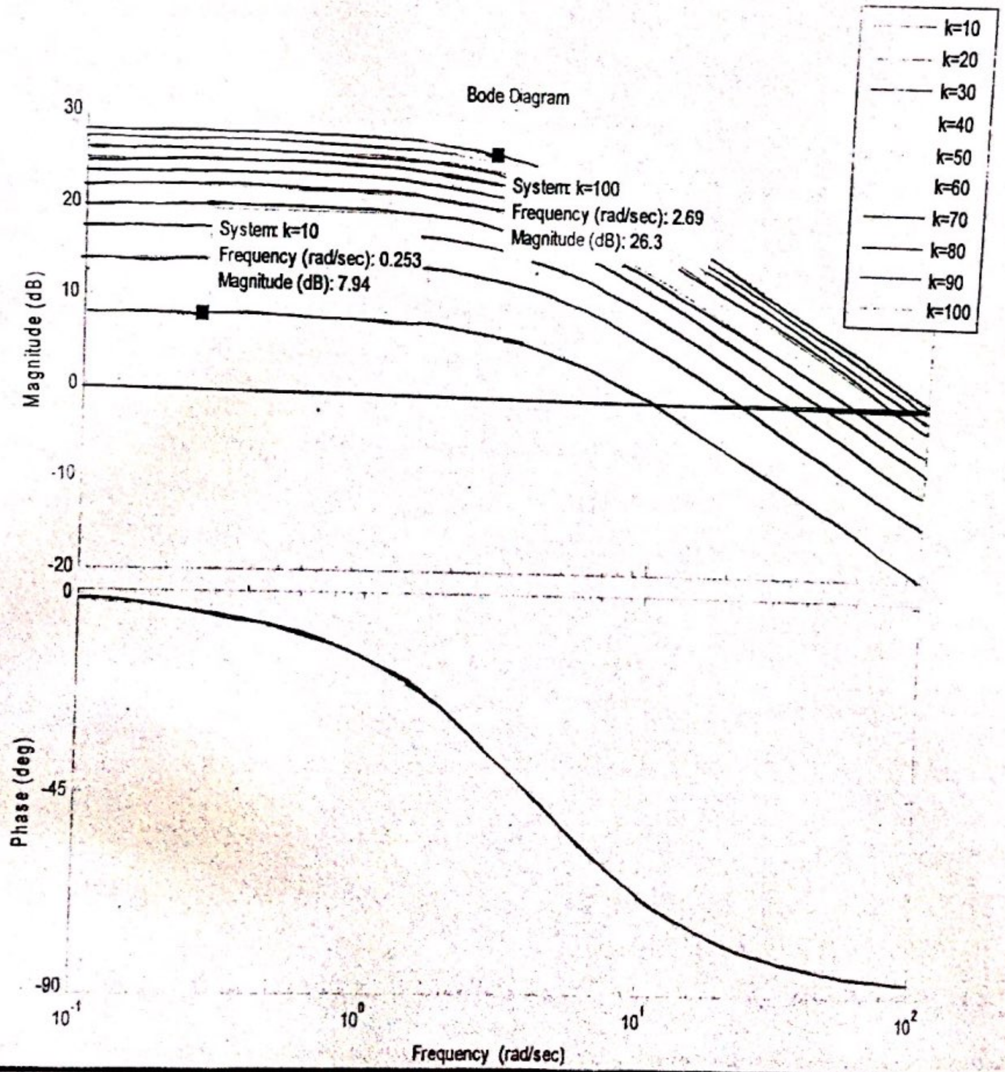




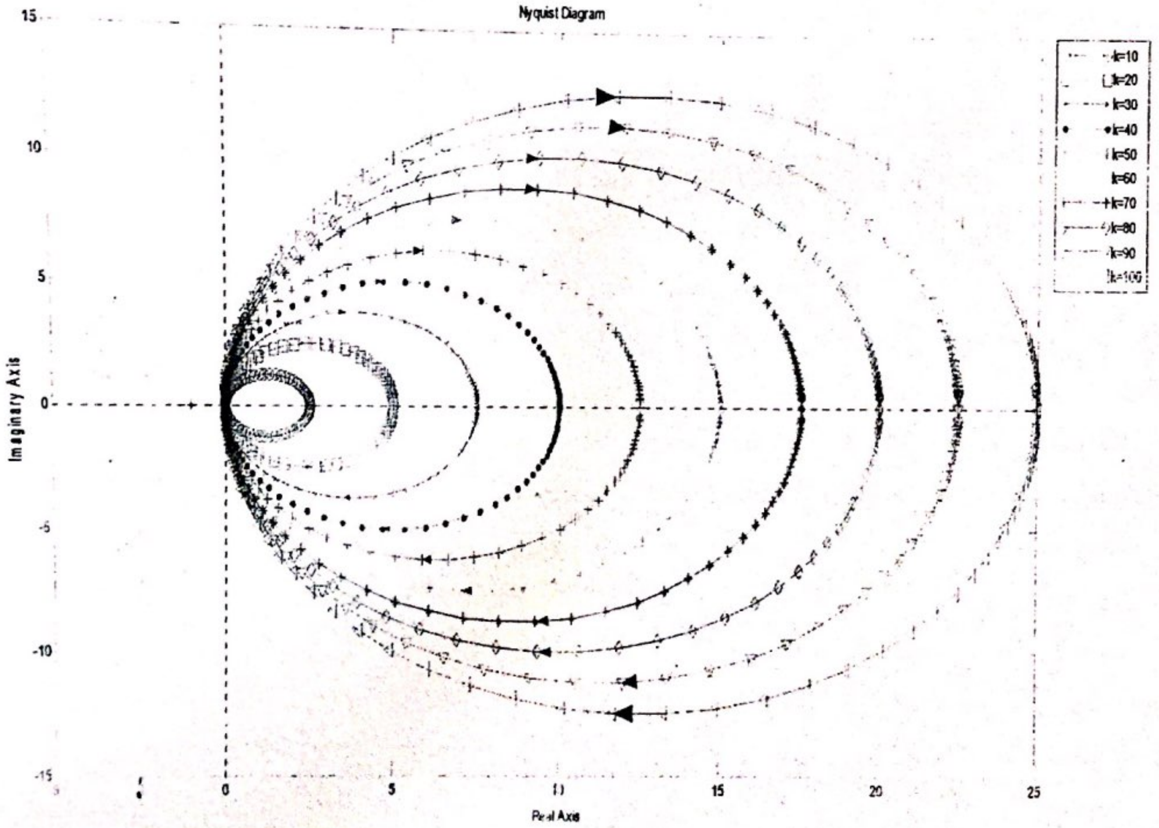
EX Sketch the bode plot and Nyquist diagram for following system with unity feedback .for K =10,20,30,40,50,60,70,80,90,100 and check the stability in each case.

$$G(s) = \frac{K}{(s+4)}$$

```
for k= 10:10:100
g= tf(k,[1 4])
bode(g)
hold on
end
legend('k=10','k=20','k=30','k=40','k=50','k=60',
'k=70','k=80','k=90','k=100')
```

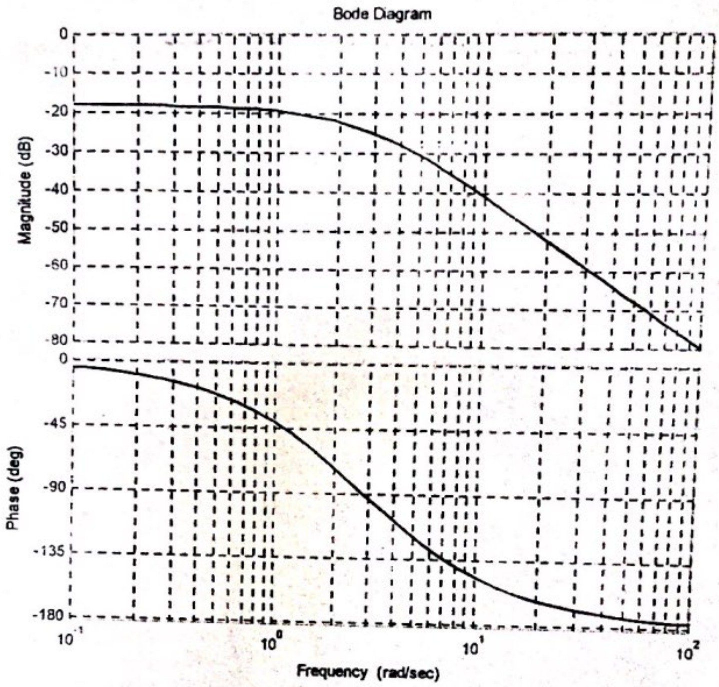


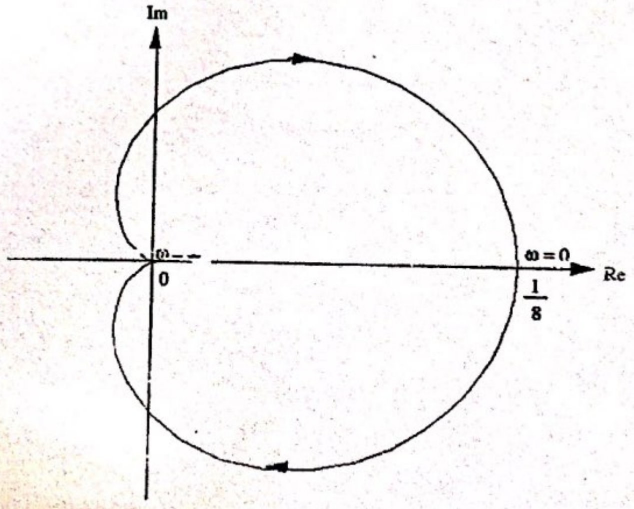
```
for k= 10:10:100
g= tf(k,[1 4])
nyquist(g)
hold on
end
legend('k=10','k=20','k=30','k=40','k=50','k=60','k=70'
,'k=80','k=90','k=100')
```



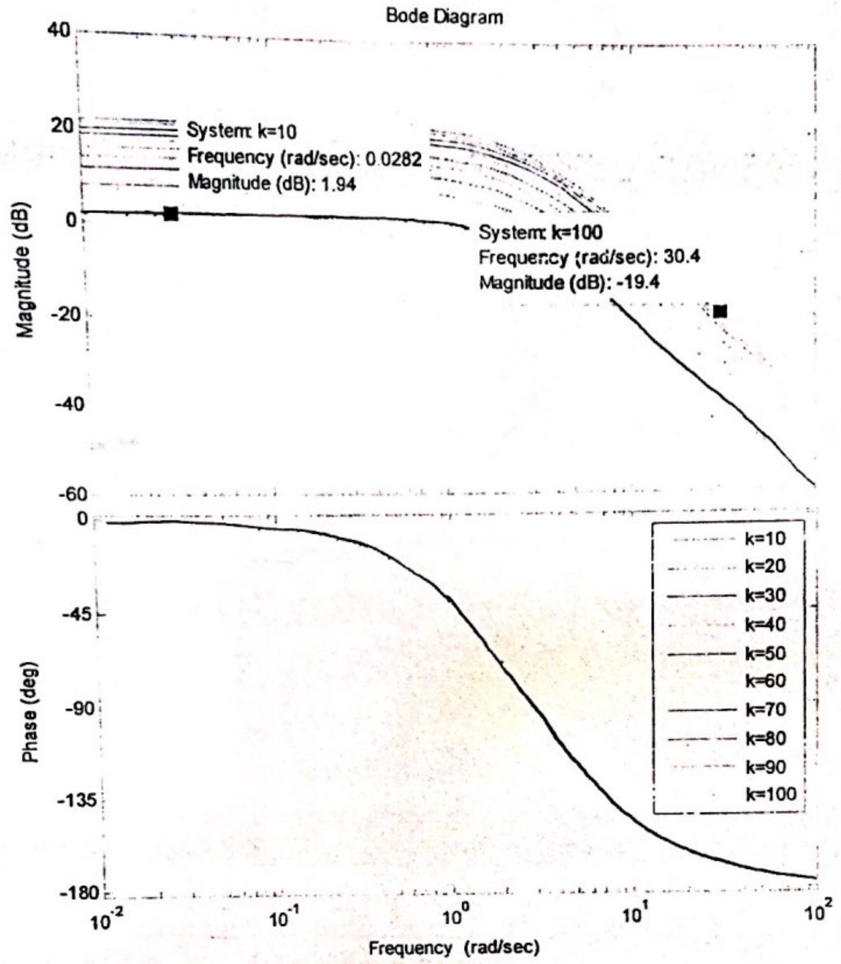
EX Sketch the bode plot and Nyquist diagram for following system with unity feedback.

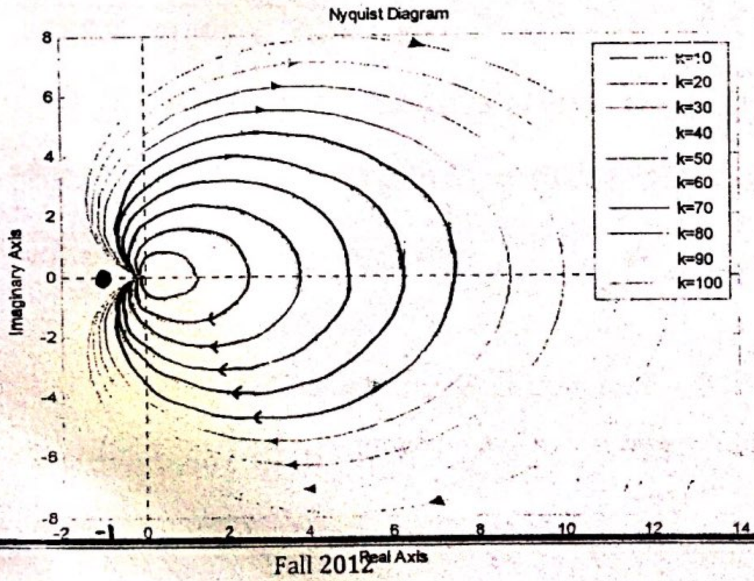
$$G(s) = \frac{1}{(s+2)(s+4)}$$





$$G(s) = \frac{K}{(s+2)(s+4)}$$





EX For the following system unity feedback.

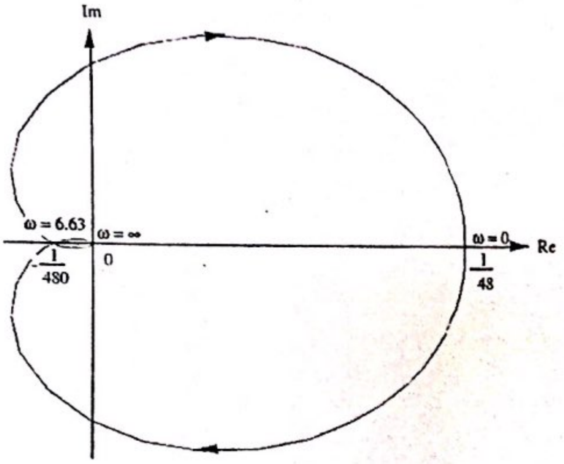
$$G(s) = \frac{k}{(s+2)(s+4)(s+6)}$$

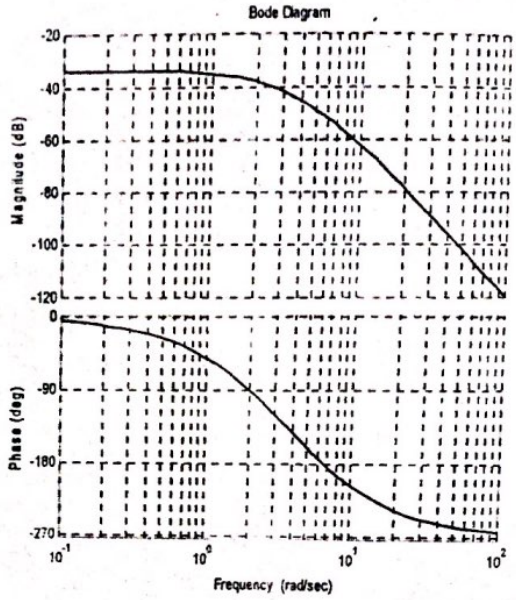
do the following:

a. Plot the bode plot and Nyquist diagram.

b. Use your Nyquist diagram to find the range of gain, K, for stability.

c. Use your Nyquist diagram to find the gain margin and the 180° frequency if k=100 .



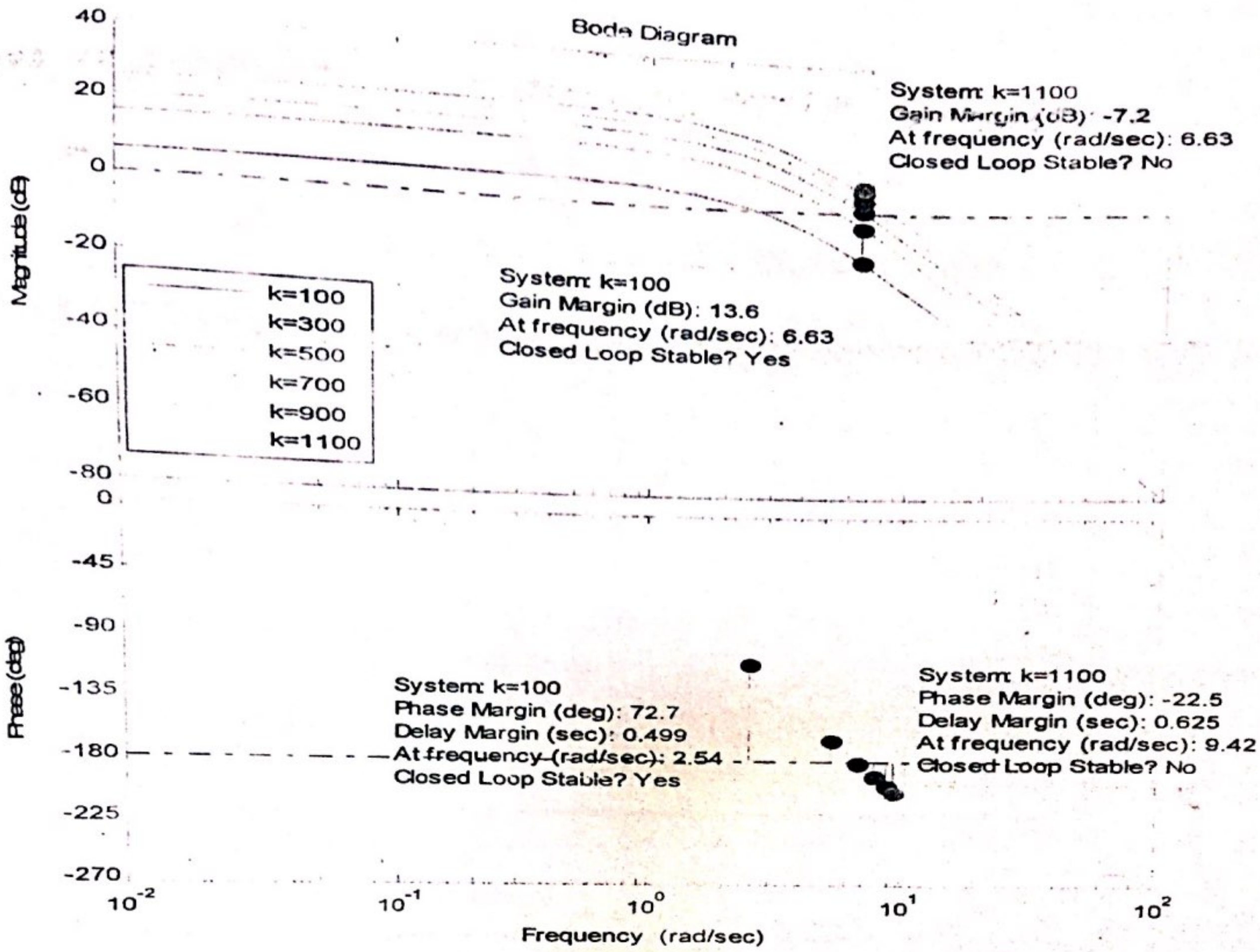


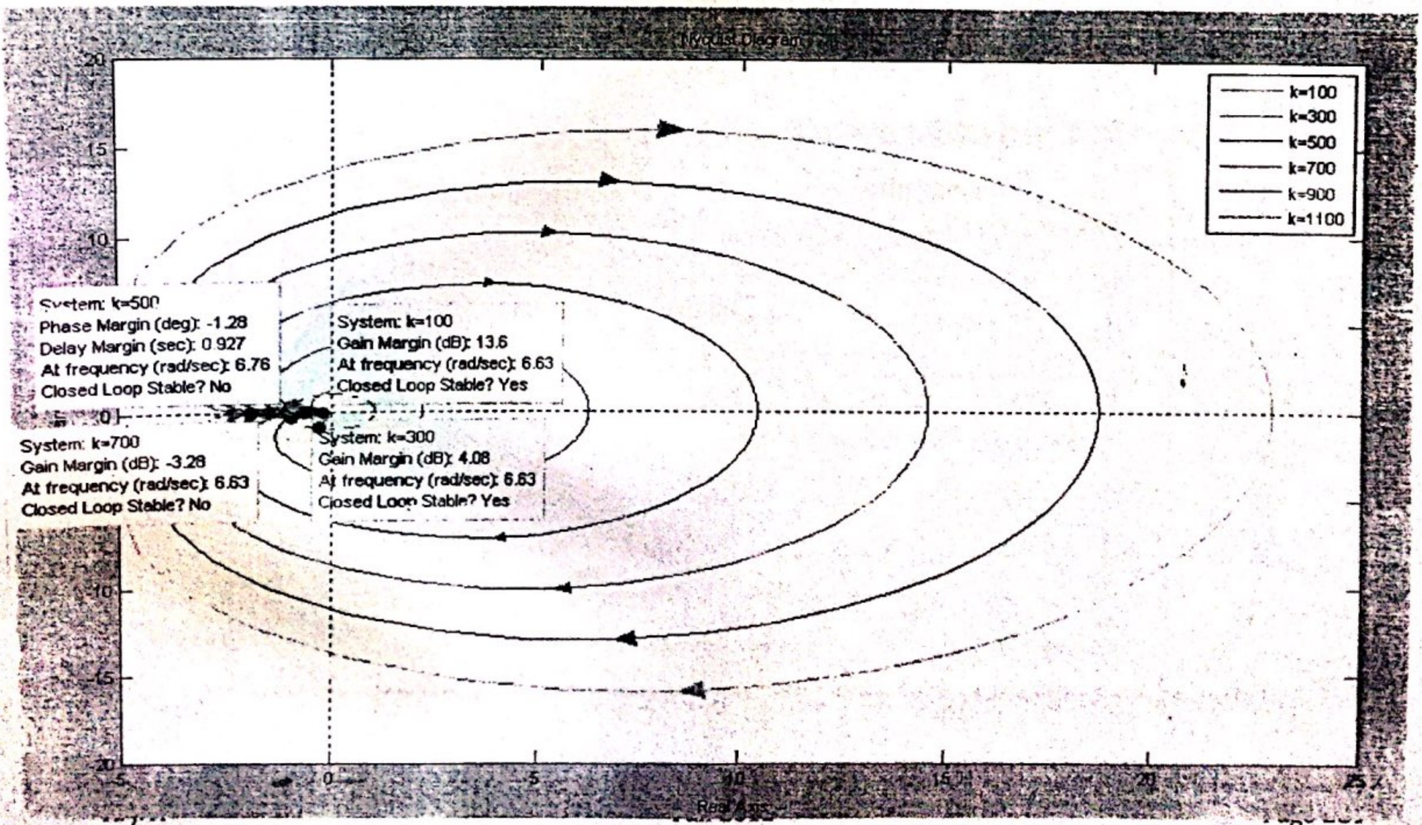
b. Substituting $j\omega$ into $G(s) = \frac{1}{(s+2)(s+4)(s+6)} = \frac{1}{s^3+12s^2+44s+48}$ and

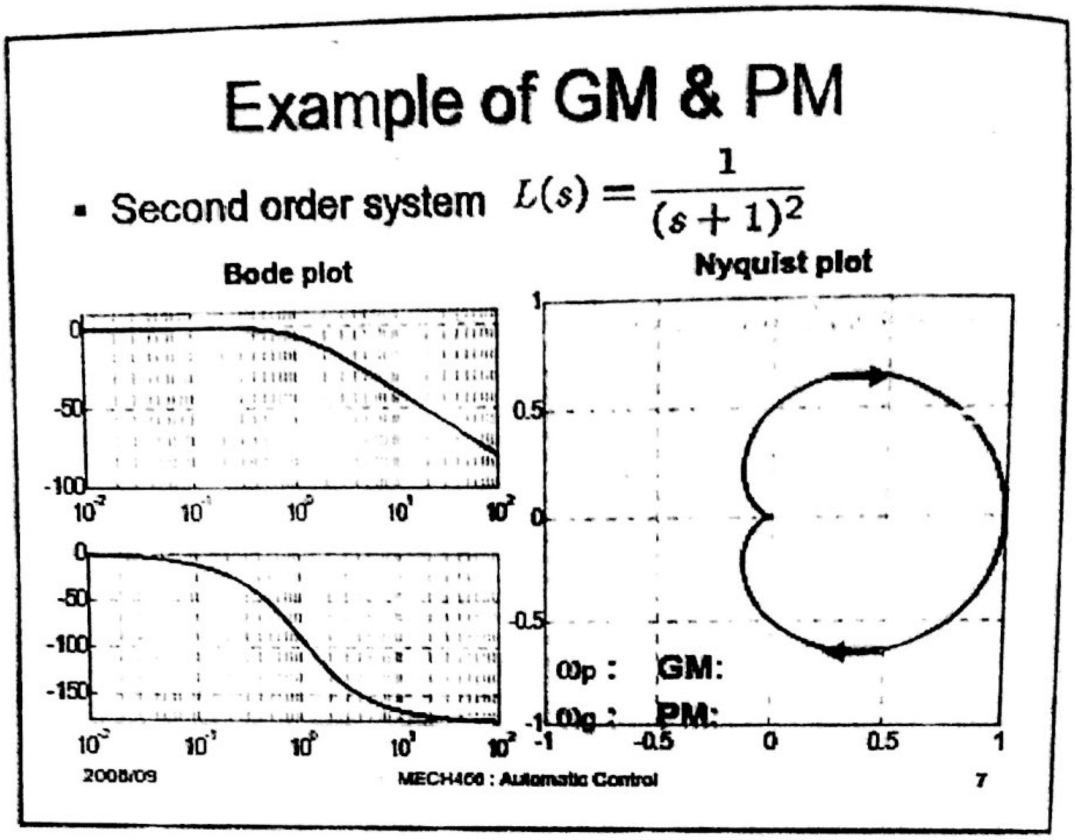
simplifying, we obtain $G(j\omega) = \frac{(48--12\omega^2) - j(44\omega - \omega^3)}{\omega^6 + 56\omega^4 + 784\omega^2 + 2304}$. The Nyquist diagram crosses the real axis when the imaginary part of $G(j\omega)$ is zero. Thus, the Nyquist diagram crosses the real axis at $\omega^2 = 44$, or $\omega = \sqrt{44} = 6.63$ rad/s. At this frequency $G(j\omega) = --\frac{1}{480}$. Thus, the system is stable for $K < 480$.

If $K = 100$, the Nyquist diagram will intersect the real axis at –100/480. Thus,

$G_M = 20\log\frac{480}{100} = 13.62$ dB. the 180⁰ frequency is 6.63 rad/s.


Bode Diagram
Magnitude (dB)
Phase (deg)
Frequency (rad/sec)
40
20
0
-20
-40
-60
-80
0
-45
-90
-135
-180
-225
-270
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
k=100
k=300
k=500
k=700
k=900
k=1100
System: k=1100
Gain Margin (dB): -7.2
At frequency (rad/sec): 6.63
Closed Loop Stable? No
System: k=100
Gain Margin (dB): 13.6
At frequency (rad/sec): 6.63
Closed Loop Stable? Yes
System: k=100
Phase Margin (deg): 72.7
Delay Margin (sec): 0.499
At frequency (rad/sec): 2.54
Closed Loop Stable? Yes
System: k=1100
Phase Margin (deg): -22.5
Delay Margin (sec): 0.625
At frequency (rad/sec): 9.42
Closed Loop Stable? No


Nyquist Diagram
Real Axis
20
15
10
0
-10
-15
-20
-5
0
5
10
15
20
25
k=100
k=300
k=500
k=700
k=900
k=1100
System: k=500
Phase Margin (deg): -1.28
Delay Margin (sec): 0.927
At frequency (rad/sec): 6.76
Closed Loop Stable? No
System: k=100
Gain Margin (dB): 13.6
At frequency (rad/sec): 6.63
Closed Loop Stable? Yes
System: k=700
Gain Margin (dB): -3.28
At frequency (rad/sec): 6.63
Closed Loop Stable? No
System: k=300
Gain Margin (dB): 4.08
At frequency (rad/sec): 6.63
Closed Loop Stable? Yes



Example of GM & PM

- Second order system $L(s) = \frac{1}{(s+1)^2}$

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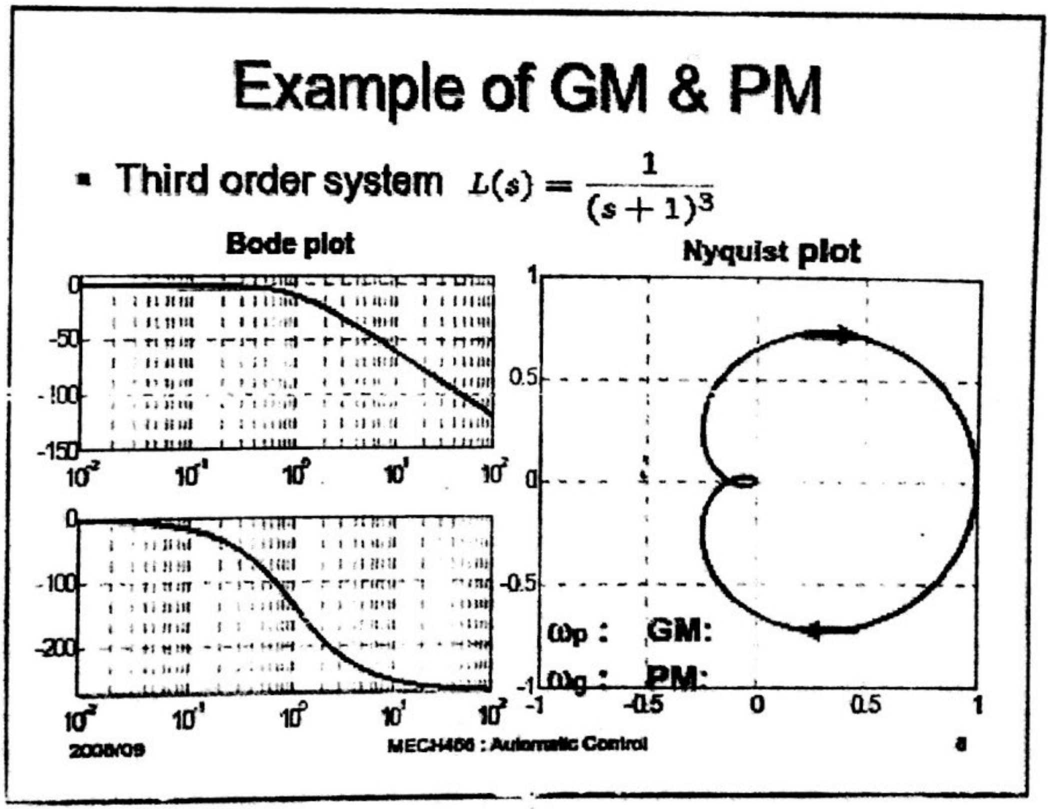

Example of GM & PM

- Third order system $L(s) = \frac{1}{(s+1)^3}$

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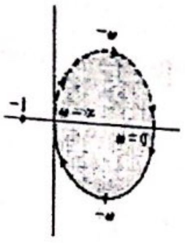
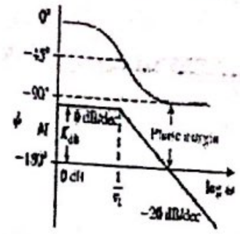
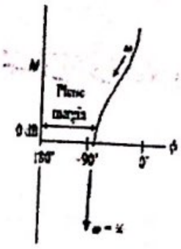
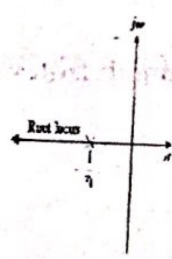
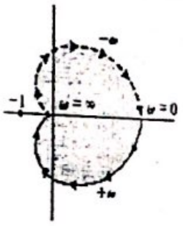
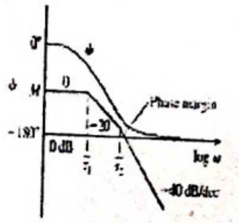
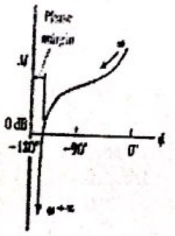
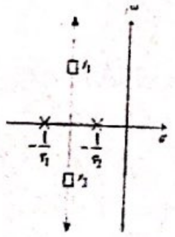
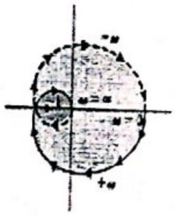
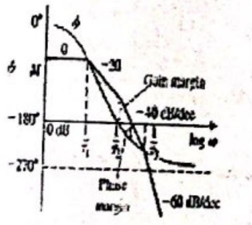
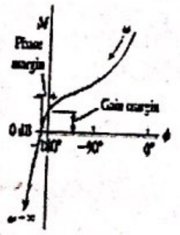
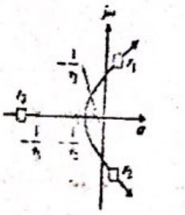
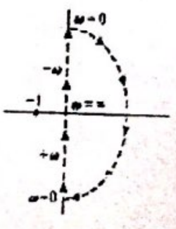
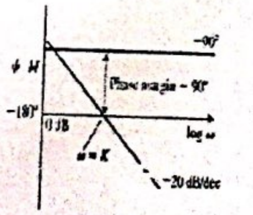
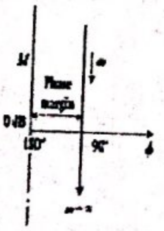
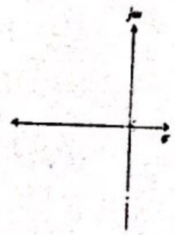
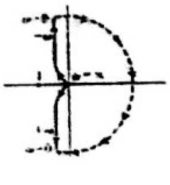
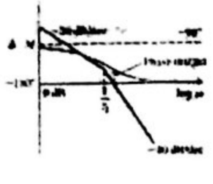
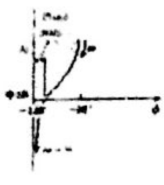
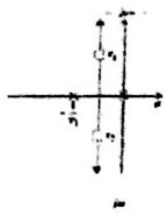
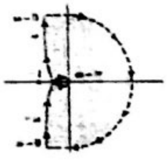
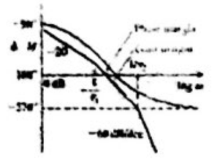
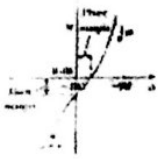
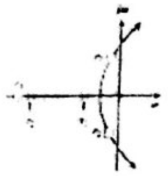
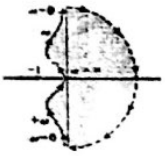
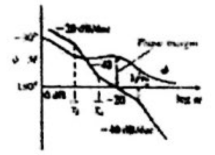
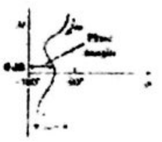
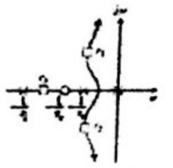
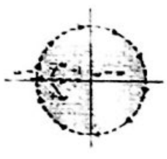
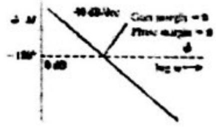
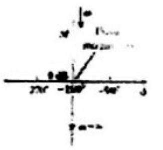
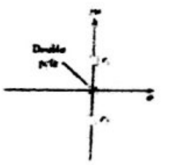
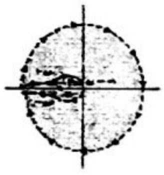
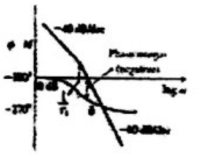
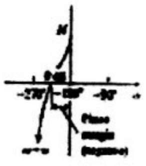
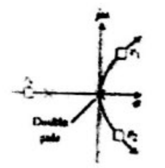
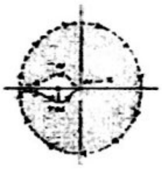
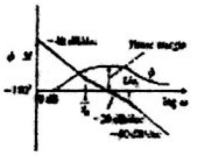
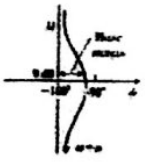
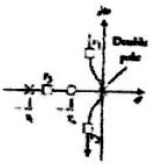
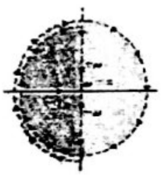
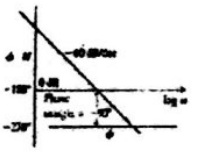
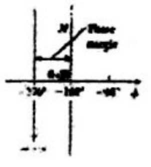
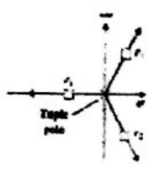
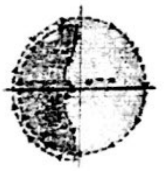
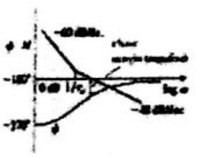
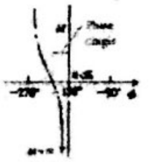
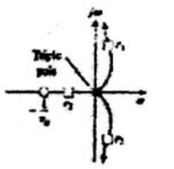
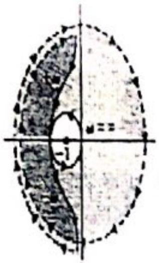
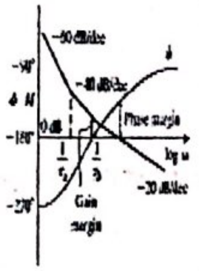
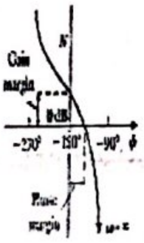
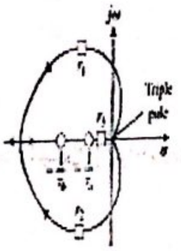
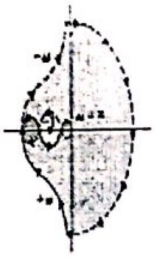
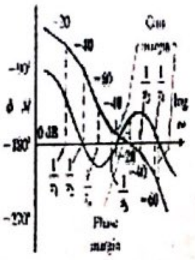
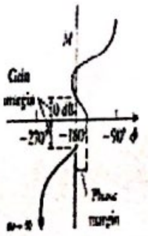
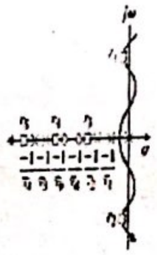
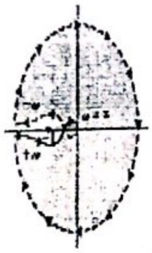
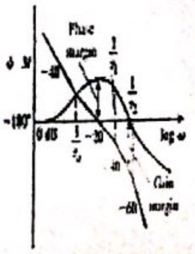
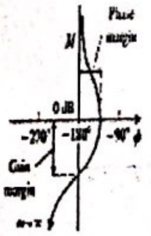
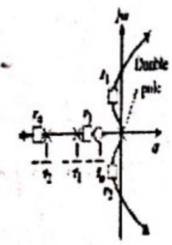
L(s)	Polar Plot	Bode Diagram	Nichols Diagram	Root Locus	Comments
1. $\frac{K}{s\tau_1 + 1}$					Stable; gain margin = ∞
2. $\frac{K}{(s\tau_1 + 1)(s\tau_2 + 1)}$					Elementary regulator; stable; gain margin = ∞
3. $\frac{K}{(s\tau_1 + 1)(s\tau_2 + 1)(s\tau_3 + 1)}$					Regulator with additional energy-storage component; unstable, but can be made stable by reducing gain
4. $\frac{K}{S}$					Ideal integrator; stable

L(s)	Polar Plot	Bode Diagram	Nichols Diagram	Root Locus	Comments
5. $\frac{K}{s(s\tau_1+1)}$					Elementary instrument servo; inherently stable; gain margin = ∞
6. $\frac{K}{s(s\tau_1+1)(s\tau_2+1)}$					Instrument servo with field control motor or power servo with elementary Ward-Leonard drive; stable as shown, but may become unstable with increased gain
7. $\frac{K(s\tau_a+1)}{(s\tau_1+1)(s\tau_2+1)}$					Elementary instrument servo; phase-lead (derivative) compensator; stable
8. $\frac{K}{s^2}$					Inherently marginally stable; must be compensated
9. $\frac{K}{s(s\tau_1-1)}$					Inherently unstable; must be compensated
10. $\frac{K(s\tau_a+1)}{s^2(s\tau_1+1)}$ $\tau_a > \tau_1$					Stable for all gains
11. $\frac{K}{s^3}$					Inherently unstable
12. $\frac{K(s\tau_a+1)}{s^3}$					Inherently unstable

(continued)

$L(s)$	Polar Plot	Bode Diagram	Nichols Diagram	Root Locus	Comments
13. $\dfrac{K(s\tau_a+1)(s\tau_b+1)}{s^3}$					Conditionally stable; becomes unstable if gain is too low
14. $\dfrac{K(s\tau_a+1)(s\tau_b+1)}{s(s\tau_1+1)(s\tau_2+1)(s\tau_3+1)(s\tau_4+1)}$					Conditionally stable; stable at low gain, becomes unstable as gain is raised, again becomes stable as gain is further increased, and becomes unstable for very high gains
15. $\dfrac{K(s\tau_a+1)}{s^2(s\tau_1+1)(s\tau_2+1)}$					Conditionally stable; becomes unstable at high gain